# THE WORKS

OF

# GEORGE BERKELEY, D.D.,

BISHOP OF CLOYNE.

INCLUDING

HIS LETTERS TO THOMAS PRIOR, Esq., DEAN GERVAIS,
MR. POPE, &c. &c.

TO WHICH IS PREFIXED

AN ACCOUNT OF HIS LIFE.

IN THIS EDITION THE LATIN ESSAYS ARE RENDERED INTO ENGLISH, AND THE "INTRODUCTION TO HUMAN KNOWLEDGE" ANNOTATED,

BY THE

REV. G. N. WRIGHT, M.A.

EDITOR OF THE WORKS OF REID AND STEWART.

IN TWO VOLUMES.

VOL. II.

LONDON:

PRINTED FOR THOMAS TEGG, CHEAPSIDE;

R. GRIFFIN AND CO., GLASGOW; T. LE MESSURIER, DUBLIN;
J. AND S. A. TEGG, SYDNEY AND HOBART TOWN.

MDCCCXLIII.

LONDON:
J. HADDON, PRINTER, CASTLE STREET, FINSBURY.

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# PASSIVE OBEDIENCE,

OR,

THE CHRISTIAN DOCTRINE OF NOT RESISTING THE SUPREME POWER,

PROVED AND VINDICATED

UPON

THE PRINCIPLES OF THE LAW OF NATURE,

IN A DISCOURSE DELIVERED AT THE COLLEGE CHAPEL.

# TO THE READER.

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That an absolute passive obedience ought not to be paid to any civil power: but that submission to government should be measured and limited by the public good of the society; and that therefore subjects may lawfully resist the supreme authority, in those cases where the public good shall plainly seem to require it: nay, that it is their duty to do so, inasmuch as they are all under an indispensable obligation to promote the common interest; these and the like notions, which I cannot help thinking pernicious to mankind and repugnant to right reason, having of late years been industriously cultivated, and set in the most advantageous lights by men of parts and learning, it seemed necessary to arm the youth of our university against them, and take care they go into the world well principled; I do not mean obstinately prejudiced in favour of a party, but from an early acquaintance with their duty, and the clear rational grounds of it, determined to such practices as may speak them good Christians and loyal subjects.

In this view, I made three discourses not many months since in the College-chapel,* which some who heard them thought it might be of use to make more public: and indeed, the false accounts that are gone abroad concerning them, have made it necessary. Accordingly I now send them into the world under the form of one entire discourse.

To conclude; as in writing these thoughts it was my endeavour to preserve that cool and impartial temper which becomes every sincere inquirer after truth, so I heartily wish they may be read with the same disposition.

* Trinity College, Dublin.

# PASSIVE OBEDIENCE,

ETC.

ROMANS XIII. 2.

"*Whosoever resisteth the power resisteth the ordinance of God.*"

I. IT is not my design to inquire into the particular nature of the government and constitution of these kingdoms; much less to pretend to determine concerning the merits of the different parties now reigning in the state. Those topics I profess to lie out of my sphere, and they will probably be thought by most men, improper to be treated of in an audience almost wholly made up of young persons, set apart from the business and noise of the world, for their more convenient instruction in learning and piety. But surely it is in no respect unsuitable to the circumstances of this place to inculcate and explain every branch of the law of nature; or those virtues and duties which are equally binding in every kingdom or society of men under heaven; and of this kind I take to be that Christian duty of not resisting the supreme power implied in my text. "Whosoever resisteth the power resisteth the ordinance of God." In treating on which words I shall observe the following method.

II. First I shall endeavour to prove, that there is an absolute, unlimited non-resistance or passive obedience due to the supreme civil power, wherever placed in any nation. Secondly, I shall inquire into the grounds and reasons of the contrary opinion. Thirdly, I shall consider the objections drawn from the pretended consequences of non-resistance to the supreme power. In handling these points I intend not to build on the authority of holy scripture, but altogether on the principles of reason common to all mankind; and that, because there are some very rational and learned men, who being verily persuaded, an absolute passive subjection to any earthly power is repugnant to right reason, can never bring themselves to admit such an interpretation of holy scripture (however natural and obvious from the words) as shall make that a part of Christian religion, which seems to them in itself manifestly absurd, and destructive of the original inherent rights of human nature.

III. I do not mean to treat of that submission which men are either in duty or prudence obliged to pay inferior or executive powers; neither shall I consider where or in what persons the supreme or legislative power is lodged in this or that government. Only thus much I shall take for granted, that there is in every civil community, somewhere or other, placed a supreme power of making laws, and enforcing the observation of them. The fulfilling of those laws, either by a punctual performance of what is enjoined in them, or, if that be inconsistent with reason or conscience, by a patient submission to whatever penalties the supreme power hath annexed to the neglect or transgression of them, is termed loyalty; as on the other hand, the making use of force and open violence, either to withstand the execution of the laws, or ward off the penalties appointed by the supreme power, is properly named rebellion. Now to make it evident, that every degree of rebellion is criminal in the subject; I shall in the first place endeavour to prove that loyalty is a natural or moral duty; and disloyalty or rebellion in the most strict and proper sense, a vice or breach of the law of nature. And secondly, I propose to show that the prohibitions of vice, or negative precepts of the law of nature, as, Thou shalt not commit adultery, Thou shalt not forswear thyself, Thou shalt not resist the supreme power, and the like, ought to be taken in a most absolute, necessary, and immutable sense: insomuch that the attainment of the greatest good, or deliverance from the greatest evil, that can befall any man or number of men in this life, may not justify the least violation of them. First then I am to show that loyalty is a moral duty, and disloyalty or rebellion in the most strict and proper sense a vice, or breach of the law of nature.

IV. Though it be a point agreed amongst all wise men, that there are certain moral rules or laws of nature, which carry with them an eternal and indispensable obligation; yet concerning the proper methods for discovering those laws, and distinguishing them from others dependent on the humour and discretion of men, there are various opinions; some direct us to look for them in the divine ideas, others in the natural inscriptions on the mind; some derive them from the authority of learned men, and the universal agreement and consent of nations. Lastly, others hold that they are only to be discovered by the deductions of reason. The three first methods must be acknowledged to labour under great difficulties, and the last has not, that I know, been any where distinctly explained, or treated of so fully as the importance of the subject doth deserve. I hope therefore it will be pardoned, if in a discourse of passive obedience, in order to lay the foundation of that duty the deeper, we make some inquiry into the origin, nature, and obligation of moral duties in general, and the criterions whereby they are to be known.

V. Self-love being a principle of all others the most universal, and the most deeply engraven in our hearts, it is natural for us to regard things as they are fitted to augment or impair our own happiness; and accordingly we denominate them good or evil. Our judgment is ever employed in distinguishing between these two, and it is the whole business of our lives to endeavour, by a proper application of our faculties, to procure the one and avoid the other. At our first coming into the world we are entirely guided by the impressions of sense, sensible pleasure being the infallible characteristic of present good, as pain is of evil. But by degrees, as we grow up in our acquaintance with the nature of things, experience informs us that present good is afterwards oft attended with a greater evil; and on the other side, that present evil is not less frequently the occasion of procuring to us a greater future good. Besides, as the nobler faculties of the human soul begin to display themselves, they discover to us goods far more excellent than those which affect the senses. Hence an alteration is wrought in our judgments; we no longer comply with the first solicitations of sense, but stay to consider the remote consequences of an action, what good may be hoped, or what evil feared from it, according to the wonted course of things. This obliges us frequently to overlook present momentary enjoyments, when they come in competition with greater and more lasting goods, though too far off, or of too refined a nature to affect our senses.

VI. But as the whole earth, and the entire duration of those perishing things contained in it, is altogether inconsiderable, or in the prophet's expressive style, "less than nothing" in respect of eternity, who sees not that every reasonable man ought so to frame his actions as that they may most effectually contribute to promote his eternal interest? And since it is a truth evident by the light of nature, that there is a sovereign, omniscient Spirit, who alone can make us for ever happy, or for ever miserable: it plainly follows that a conformity to his will, and not any prospect of temporal advantage, is the sole rule whereby every man who acts up to the principles of reason must govern and square his actions. The same conclusion doth likewise evidently result from the relation which God bears to his creatures. God alone is maker and preserver of all things. He is therefore with the most undoubted right the great legislator of the world; and mankind are by all the ties of duty, no less than interest, bound to obey his laws.

VII. Hence we should above all things endeavour to trace out the divine will, or the general design of Providence with regard to mankind, and the methods most directly tending to the accomplishment of that design, and this seems the genuine and proper way for discovering the laws of nature. For laws being rules directive of our actions to the end intended by the legislator, in order to attain the knowledge of God's laws, we ought first to inquire what that end is, which he designs should be carried on by human actions. Now, as God is a being of infinite goodness, it is plain the end he proposes is good. But God enjoying in himself all possible perfection, it follows that it is not his own good, but that of his creatures. Again, the moral actions of men are entirely terminated within themselves, so as to have no influence on the other orders of intelligences or reasonable creatures: the end therefore to be procured by them, can be no other than the good of men. But as nothing in a natural state can entitle one man more than another to the favour of God, except only moral goodness, which consisting in a conformity to the laws of God, doth presuppose the being of such laws, and law ever supposing an end, to which it guides our actions, it follows that antecedent to the end proposed by God, no distinction can be conceived between men; that end therefore itself, or general design of Providence, is not determined or limited by any respect of persons: it is not therefore the private good of this or that man, nation, or age, but the general well-being of all men, of all nations, of all ages of the world, which God designs should be procured by the concurring actions of each individual. Having thus discovered the great end to which all moral obligations are subordinate; it remains, that we inquire what methods are necessary for the obtaining that end.

VIII. The well-being of mankind must necessarily be carried on one of these two ways: either first, without the injunction of any certain universal rules of morality, only by obliging every one upon each particular occasion to consult the public good, and always to do that which to him shall seem, in the present time and circumstances, most to conduce to it. Or secondly, by enjoining the observation of some determinate, established laws, which, if universally practised, have from the nature of things an essential fitness to procure the well-being of mankind; though in their particular application, they are sometimes, through untoward accidents and the perverse irregularity of human wills, the occasions of great sufferings and misfortunes, it may be, to very many good men. Against the former of these methods there lie several strong objections. For brevity I shall mention only two.

IX. First, it will thence follow, that the best men for want of judgment, and the wisest for want of knowing all the hidden circumstances and consequences of an action, may very often be at a loss how to behave themselves; which they would not be, in case they judged of each action by comparing it with some particular precept, rather than by examining the good or evil which in that single instance it tends to procure: it being far more easy to judge with certainty, whether such or such an action be a transgression of this or that precept, than whether it will be attended with more good or ill consequences. In short, to calculate the events of each particular action is impossible, and though it were not, would yet take up too much time to be of use in the affairs of life. Secondly, if that method be observed, it will follow that we can have no sure standard, to which comparing the actions of another, we may pronounce them good or bad, virtues or vices. For since the measure and rule of every good man's actions is supposed to be nothing else, but his own private, disinterested opinion, of what makes most for the public good at that juncture: and since this opinion must unavoidably in different men, from their particular views and circumstances, be very different: it is impossible to know, whether any one instance of parricide or perjury, for example, be criminal. The man may have had his reasons for it, and that which in me would have been a heinous sin, may be in him a duty. Every man's particular rule is buried in his own breast, invisible to all but himself, who therefore can only tell whether he observes it or no. And since that rule is fitted to particular occasions, it must ever change as they do: hence it is not only various in different men, but in one and the same man at different times.

X. From all which it follows, there can be no harmony or agreement between the actions of good men: no apparent steadiness or consistency of one man with himself, no adhering to principles: the best actions may be condemned, and the most villanous meet with applause. In a word, there ensues the most horrible confusion of vice and virtue, sin and duty, that can possibly be imagined. It follows therefore that the great end to which God requires the concurrence of human actions, must of necessity be carried on by the second method proposed, namely, the observation of certain, universal, determinate rules or moral precepts, which in their own nature have a necessary tendency to promote the well-being of the sum of mankind, taking in all nations and ages, from the beginning to the end of the world.

XI. Hence upon an equal comprehensive survey of the general nature, the passions, interests, and mutual respects of mankind; whatsoever practical proposition doth to right reason evidently appear to have a necessary connexion with the universal well-being included in it, is to be looked upon as enjoined by the will of God. For he that willeth the end, doth will the necessary means conducive to that end; but it hath been shown, that God willeth the universal well-being of mankind should be promoted by the concurrence of each particular person; therefore every such practical proposition, necessarily tending thereto, is to be esteemed a decree of God, and is consequently a law to man.

XII. These propositions are called laws of nature, because they are universal, and do not derive their obligation from any civil sanction, but immediately from the author of nature himself. They are said to be stamped on the mind, to be engraven on the tables of the heart, because they are well known to mankind, and suggested and inculcated by conscience. Lastly, they are termed eternal rules of reason, because they necessarily result from the nature of things, and may be demonstrated by the infallible deductions of reason.

XIII. And notwithstanding that these rules are too often, either by the unhappy concurrence of events, or more especially by the wickedness of perverse men, who will not conform to them, made accidental causes of misery to those good men who do; yet this doth not vacate their obligation: they are ever to be esteemed the fixed unalterable standards of moral good and evil; no private interest, no love of friends, no regard to the public good, should make us depart from them. Hence when any doubt arises concerning the morality of an action, it is plain, this cannot be determined by computing the public good, which in that particular case it is attended with, but only by comparing it with the eternal law of reason. He who squares his actions by this rule, can never do amiss, though thereby he should bring himself to poverty, death, or disgrace: no, not though he should involve his family, his friends, his country in all those evils, which are accounted the greatest, and most insupportable to human nature. Tenderness and benevolence of temper are often motives to the best and greatest actions; but we must not make them the sole rule of our actions; they are passions rooted in our nature, and like all other passions must be restrained and kept under, otherwise they may possibly betray us into as great enormities, as any other unbridled lust. Nay, they are more dangerous than other passions, insomuch as they are more plausible, and apt to dazzle and corrupt the mind, with the appearance of goodness and generosity.

XIV. For the illustration of what has been said, it will not be amiss, if from the moral we turn our eyes on the natural world. *Homo ortus est* (says Balbus in Cicero*) *ad mundum contemplandum et imitandum:* and surely it is not possible for free intellectual agents to propose a nobler pattern for their imitation than nature, which is nothing else but a series of free actions, produced by the best and wisest agent. But it is evident that those actions are not adapted to particular views, but all conformed to certain general rules, which being collected from observation, are by philosophers termed laws of nature. And these indeed are excellently suited to promote the general well-being of the creation: but what from casual combinations of events, and what from the voluntary motions of animals, it often falls out that the natural good, not only of private men

* De Natura Deorum, lib. ii.

but of entire cities and nations, would be better promoted by a particular suspension, or contradiction, than an exact observation of those laws. Yet for all that, nature still takes its course; nay, it is plain that plagues, famines, inundations, earthquakes, with an infinite variety of pains and sorrows; in a word, all kinds of calamities, public and private, do arise from a uniform, steady observation of those general laws, which are once established by the author of nature, and which he will not change or deviate from upon any of those accounts, how wise or benevolent soever it may be thought by foolish men to do so. As for the miracles recorded in scripture, they were always wrought for confirmation of some doctrine or mission from God, and not for the sake of the particular natural goods, as health or life, which some men might have reaped from them. From all which it seems sufficiently plain, that we cannot be at a loss which way to determine, in case we think God's own methods the properest to obtain his ends, and that it is our duty to copy after them, so far as the frailty of our nature will permit.

XV. Thus far, in general, of the nature and necessity of moral rules, and the criterion or mark whereby they may be known. As for the particulars, from the foregoing discourse, the principal of them may, without much difficulty, be deduced. It hath been shown, that the law of nature is a system of such rules or precepts, as that if they be all of them, at all times, in all places, and by all men observed, they will necessarily promote the well-being of mankind, so far as it is attainable by human actions. Now let any one who hath the use of reason take but an impartial survey of the general frame and circumstances of human nature, and it will appear plainly to him, that the constant observation of truth, for instance, of justice, and chastity, hath a necessary connexion with their universal well-being; that therefore they are to be esteemed virtues or duties, and that, *thou shalt not forswear thyself, thou shalt not commit adultery, thou shalt not steal,* are so many unalterable moral rules, which, to violate in the least degree, is vice or sin. I say, the agreement of these particular practical propositions, with the definition or criterion premised, doth so clearly result from the nature of things, that it were a needless digression in this place to enlarge upon it. And from the same principle, by the very same reasoning, it follows, that loyalty is a moral virtue, and, *thou shalt not resist the supreme power,* a rule or law of nature, the least breach whereof hath the inherent stain of moral turpitude.

XVI. The miseries inseparable from a state of anarchy are easily imagined. So insufficient is the wit or strength of any single man, either to avert the evils, or procure the blessings of life, and so apt are the wills of different persons to contradict and thwart each other, that it is absolutely necessary, several independent powers be combined together, under the direction (if I may so speak) of one and the same will, I mean the law of the society. Without this there is no politeness, no order, no peace among men, but the world is one great heap of misery and confusion; the strong as well as the weak, the wise as well as the foolish, standing on all sides exposed to all those calamities, which man can be liable to in a state where he has no other security, than the not being possessed of any thing which may raise envy or desire in another. A state by so much more ineligible than that of brutes, as a reasonable creature hath a greater reflection and foresight of miseries than they. From all which it plainly follows, that loyalty, or submission to the supreme authority, hath, if universally practised in conjunction with all other virtues, a necessary connexion with the well-being of the whole sum of mankind; and by consequence, if the criterion we have laid down be true, it is, strictly speaking, a moral duty, or branch of natural religion. And, therefore, the least degree of rebellion is, with the utmost strictness and propriety, a *sin:* not only in Christians, but also in those who have the light of reason alone for their guide. Nay, upon a thorough and impartial view, this submission will, I think, appear one of the very first and fundamental laws of nature, inasmuch as it is civil government which ordains and marks out the various relations between men, and regulates property, thereby giving scope and laying a foundation for the exercise of all other duties. And, in truth, whoever considers the condition of man, will scarce conceive it possible that the practice of any one moral virtue should obtain, in the naked, forlorn state of nature.

XVII. But since it must be confessed, that in all cases our actions come not within the direction of certain fixed moral rules, it may possibly be still questioned, whether obedience to the supreme power be not one of those exempted cases, and, consequently, to be regulated by the prudence and discretion of every single person, rather than adjusted to the rule of absolute non-resistance. I shall therefore endeavour to make it yet more plain, that, *thou shalt not resist the supreme power*, is an undoubted precept of morality; as will appear from the following considerations. First, then, submission to government is a point *important* enough to be established by a moral rule. Things of insignificant and trifling concern, are, for that very reason, exempted from the rules of morality. But government, on which so much depend the peace, order, and well-being of mankind, cannot surely be thought of too small importance to be secured and guarded by a moral rule. Government, I say, which is itself the principal source under heaven, of those particular advantages, for the procurement and conservation whereof, several unquestionable moral rules were prescribed to men.

XVIII. Secondly, obedience to government is a case *uni-*

*versal* enough to fall under the direction of a law of nature. Numberless rules there may be for regulating affairs of great concernment, at certain junctures, and to some particular persons or societies, which, notwithstanding, are not to be esteemed moral or natural laws, but may be either totally abrogated or dispensed with; because the private ends they were intended to promote, respect only some particular persons, as engaged in relations not founded in the general nature of man, who, on various occasions, and in different postures of things, may prosecute their own designs by different measures, as in human prudence shall seem convenient. But what relation is there more extensive and universal than that of subject and law? This is confined to no particular age or climate, but universally obtains, at all times, and in all places, wherever men live in a state exalted above that of brutes. It is therefore evident, that the rule, forbidding resistance to the law or supreme power, is not, upon pretence of any defect in point of *universality*, to be excluded from the number of the laws of nature.

XIX. Thirdly, there is another consideration, which confirms the necessity of admitting this rule for a moral or natural law; namely, because the case it regards is of too nice and difficult a nature to be left to the judgment and determination of each private person. Some cases there are so plain and obvious to judge of, that they may safely be trusted to the prudence of every reasonable man; but in all instances, to determine whether a civil law is fitted to promote the public interest; or whether submission or resistance will prove most advantageous in the consequence; or when it is, that the general good of a nation may require an alteration of government, either in its form, or in the hands which administer it: these are points too arduous and intricate, and which require too great a degree of parts, leisure, and liberal education, as well as disinterestedness and thorough knowledge in the particular state of a kingdom, for every subject to take upon him the determination of them. From which it follows, that upon this account also, non-resistance, which, in the main, nobody can deny to be a most profitable and wholesome duty, ought not to be limited by the judgment of private persons to particular occasions, but esteemed a most sacred law of nature.

XX. The foregoing arguments do, I think, make it manifest, that the precept against rebellion is on a level with other moral rules. Which will yet further appear from this fourth and last consideration. It cannot be denied, that right reason doth require some common stated rule or measure, whereby subjects ought to shape their submission to the supreme power; since any clashing or disagreement in this point must unavoidably tend to weaken and dissolve the society. And it is unavoidable, that there should be great clashing, where it is left to the breast of each individual to suit his fancy with a different measure of obedience. But this common stated measure must be either the general precept forbidding resistance, or else the public good of the whole nation: which last, though it is allowed to be in itself something certain and determinate; yet, forasmuch as men can regulate their conduct only by what appears to them, whether in truth it be what it appears or no; and since the prospects men form to themselves of a country's public good, are commonly as various as its landscapes, which meet the eye in several situations: it clearly follows, that to make the public good the rule of obedience, is in effect not to establish any determinate, agreed, common measure of loyalty, but to leave every subject to the guidance of his own particular mutable fancy.

XXI. From all which arguments and considerations it is a most evident conclusion, that the law prohibiting rebellion is in strict truth a law of nature, universal reason, and morality. But to this, it will perhaps be objected by some, that whatever may be concluded with regard to resistance, from the tedious deductions of reason, yet there is I know not what turpitude and deformity in some actions, which at first blush shows them to be vicious; but they, not finding themselves struck with such a sensible and immediate horror at the thought of rebellion, cannot think it on a level with other crimes against nature. To which I answer, that it is true, there are certain natural antipathies implanted in the soul, which are ever the most lasting and insurmountable; but as custom is a second nature, whatever aversions are from our early childhood continually infused into the mind, give it so deep a stain as is scarce to be distinguished from natural complexion. And as it doth hence follow, that to make all the inward horrors of soul pass for infallible marks of sin, were the way to establish error and superstition in the world: so, on the other hand, to suppose all actions lawful, which are unattended with those starts of nature, would prove of the last dangerous consequence to virtue and morality. For these pertaining to us as men, we must not be directed in respect of them by any emotions in our blood and spirits, but by the dictates of sober and impartial reason. And if there be any, who find they have a less abhorrence of rebellion than of other villanies, all that can be inferred from it is, that this part of their duty was not so much reflected on, or so early and frequently inculcated into their hearts, as it ought to have been. Since without question there are other men who have as thorough an aversion for that, as for any other crime.*

* "Il disait ordinairement qu'il avait un aussi grand eloignement pour ce peché là que pour assassiner le monde, ou pour voler sur les grands chemins, et qu'enfin il n'y avait rien qui fut plus contraire à son naturel." He (M. Pascal) used to say he had as great an abhorrence of rebellion as of murder, or robbing on the highway, and that there was nothing more shocking to his nature. Vie de M. Pascal, page 44.

XXII. Again, it will probably be objected, that submission to government differs from moral duties, in that it is founded in a contract, which upon the violation of its conditions doth of course become void, and in such case rebellion is lawful: it hath not therefore the nature of a sin or crime, which is in itself absolutely unlawful, and must be committed on no pretext whatsoever. Now, passing over all inquiry and dispute concerning the first obscure rise of government, I observe its being founded on a contract may be understood in a twofold sense: either, first, that several independent persons finding the unsufferable inconvenience of a state of anarchy, where every one was governed by his own will, consented and agreed together to pay an absolute submission to the decrees of some certain legislative; which, though sometimes they may bear hard on the subject, yet must surely prove easier to be governed by, than the violent humours and unsteady opposite wills of a multitude of savages. And in case we admit such a compact to have been the original foundation of civil government, it must even on that supposition be held sacred and inviolable.

XXIII. Or secondly, it is meant that subjects have contracted with their respective sovereigns or legislators, to pay, not an absolute, but conditional and limited submission to their laws, that is, upon condition, and so far forth as the observation of them shall contribute to the public good: reserving still to themselves a right of superintending the laws, and judging whether they are fitted to promote the public good or no; and (in case they or any of them think it needful) of resisting the higher powers, and changing the whole frame of government by force: which is a right that all mankind, whether single persons or societies, have over those that are deputed by them. But in this sense a contract cannot be admitted for the ground and measure of civil obedience, except one of these two things be clearly shown: either, first, that such a contract is an express known part of the fundamental constitution of a nation, equally allowed and unquestioned by all as the common law of the land: or, secondly, if it be not express, that it is at least necessarily implied in the very nature or notion of civil polity, which supposes it is a thing manifestly absurd, that a number of men should be obliged to live under an unlimited subjection to civil law, rather than continue wild and independent of each other. But to me it seems most evident, that neither of those points will ever be proved.

XXIV. And till they are proved beyond all contradiction, the doctrine built upon them ought to be rejected with detestation. Since to represent the higher powers as deputies of the people, manifestly tends to diminish that awe and reverence, which all good men should have for the laws and government of their country. And to speak of a conditioned, limited loyalty, and I know not what vague and undetermined contracts, is a most effectual means to loosen the bands of civil society; than which nothing can be of more mischievous consequence to mankind. But after all, if there be any man, who either cannot or will not see the absurdity and perniciousness of those notions, he would, I doubt not, be convinced with a witness, in case they should once become current, and every private man take it in his head to believe them true, and put them in practice.

XXV. But there still remains an objection, which hath the appearance of some strength against what has been said. Namely, that whereas civil polity is a thing entirely of human institution, it seems contrary to reason to make submission to it part of the law of nature, and not rather of the civil law. For how can it be imagined that nature should dictate or prescribe a natural law about a thing, which depends on the arbitrary humour of men, not only as to its kind or form, which is very various and mutable, but even as to its existence; there being nowhere to be found a civil government set up by nature. In answer to this I observe first, that most moral precepts do presuppose some voluntary actions, or pacts of men, and are nevertheless esteemed laws of nature. Property is assigned, the signification of words ascertained, and matrimony contracted by the agreement and consent of mankind; and for all that it is not doubted, whether theft, falsehood, and adultery, be prohibited by the law of nature. Loyalty, therefore, though it should suppose and be the result of human institutions, may, for all that, be of natural obligation. I say, secondly, that, notwithstanding particular societies are formed by men, and are not in all places alike, as things esteemed natural are wont to be, yet there is implanted in mankind a natural tendency or disposition to a social life. I call it natural because it is universal, and because it necessarily results from the differences which distinguish man from beast: the peculiar wants, appetites, faculties, and capacities of man, being exactly calculated and framed for such a state, insomuch that, without it, it is impossible he should live in a condition in any degree suitable to his nature. And since the bond and cement of society is a submission to its laws, it plainly follows, that this duty hath an equal right with any other to be thought a law of nature. And surely that precept which enjoins obedience to civil laws, cannot itself, with any propriety, be accounted a civil law; it must therefore either have no obligation at all on the conscience, or, if it hath, it must be derived from the universal voice of nature and reason.

XXVI. And thus the first point proposed seems clearly made out: namely, that loyalty is a virtue or moral duty; and disloyalty or rebellion, in the most strict and proper sense, a vice or crime against the law of nature. We are now come to the second point, which was to show, that the prohibitions of vice, or negative precepts of morality, are to be taken in a most absolute, necessary, and immutable sense; insomuch that the attainment of the greatest good, or deliverance from the greatest evil, that can befall any man or number of men in this life, may not justify the least violation of them. But in the first place, I shall explain the reason of distinguishing between positive and negative precepts, the latter only being included in this general proposition. Now the ground of that distinction may be resolved into this; namely, that very often, either through the difficulty or number of moral actions, or their inconsistence with each other, it is not possible for one man to perform several of them at the same time; whereas it is plainly consistent and possible that any man should, at the same time, abstain from all manner of positive actions whatsoever. Hence it comes to pass, that prohibitions or negative precepts must by every one, in all times and places, be all actually observed: whereas those which enjoin the doing of an action, allow room for human prudence and discretion in the execution of them: it for the most part depending on various accidental circumstances; all which ought to be considered, and care taken that duties of less moment do not interfere with and hinder the fulfilling of those which are more important. And for this reason, if not the positive laws themselves, at least the exercise of them, admits of suspension, limitation, and diversity of degrees. As to the indispensableness of the negative precepts of the law of nature, I shall in its proof offer two arguments; the first from the nature of the thing, and the second from the imitation of God in his government of the world.

XXVII. First then, from the nature of the thing it hath been already shown that the great end of morality can never be carried on, by leaving each particular person to promote the public good, in such a manner as he shall think most convenient, without prescribing certain determinate, universal rules to be the common measure of moral actions: and, if we allow the necessity of these, and at the same time think it lawful to transgress them, whenever the public good shall seem to require it, what is this but in words indeed to enjoin the observation of moral rules, but in effect to leave every one to be guided by his own judgment? than which nothing can be imagined more pernicious and destructive to mankind, as hath been already proved. Secondly, this same point may be collected from the example set us by the author of nature, who, as we have above observed, acts according to certain fixed laws, which he will not transgress upon the account of accidental evils arising from them. Suppose a prince, on whose life the welfare of a kingdom depends, to fall down a precipice, we have no reason to think that the universal law of gravitation would be suspended in that case. The like may be said of all other laws of nature, which we do not find to admit of exceptions on particular accounts.

XXVIII. And as, without such a steadiness in nature we should soon, instead of this beautiful frame, see nothing but a disorderly and confused chaos: so if once it become current, that the moral actions of men are not to be guided by certain definite, inviolable rules, there will be no longer found that beauty, order, and agreement, in the system of rational beings, or moral world, which will then be all covered over with darkness and violence. It is true he who stands close to a palace can hardly make a right judgment of the architecture and symmetry of its several parts, the nearer ever appearing disproportionably great. And if we have a mind to take a fair prospect of the order and general well-being which the inflexible laws of nature and morality derive on the world, we must, if I may so say, go out of it, and imagine ourselves to be distant spectators of all that is transacted and contained in it; otherwise we are sure to be deceived by the too near view of the little present interests of ourselves, our friends, or our country. The right understanding of what hath been said will, I think, afford a clear solution to the following difficulties.

XXIX. First, it may perhaps seem to some that in consequence of the foregoing doctrine, men will be left to their own private judgments as much as ever. For, first, the very being of the laws of nature; secondly, the criterion whereby to know them; and, thirdly, the agreement of any particular precept with that criterion, are all to be discovered by reason and argumentation, in which every man doth necessarily judge for himself: hence upon that supposition there is place for as great confusion, unsteadiness, and contrariety of opinions and actions, as upon any other. I answer, that however men may differ, as to what were most proper and beneficial to the public to be done or omitted on particular occasions, when they have for the most part narrow and interested views; yet in general conclusions, drawn from an equal and enlarged view of things, it is not possible there should be so great, if any, disagreement at all amongst candid, rational inquirers after truth.

XXX. Secondly, the most plausible pretence of all against the doctrine we have premised concerning a rigid, indispensable observation of moral rules, is that which is founded on the consideration of the public weal: for since the common good of mankind is confessedly the end which God requires should be promoted by the free actions of men, it may seem to follow that all good men ought ever to have this in view, as the great mark to which all their endeavours should be directed; if therefore in any particular case a strict keeping to the moral rule shall prove manifestly inconsistent with the public good, it may be thought agreeable to the will of God that in that case the rule does restrain an honest disinterested person from acting for that end to which the rule itself was ordained. For it is an axiom that the end is more excellent than the means, which, deriving their goodness from the end, may not come in competition with it.

XXXI. In answer to this, let it be observed, that nothing is a law merely because it conduceth to the public good, but because it is decreed by the will of God, which alone can give the sanction of a law of nature to any precept: neither is any thing, how expedient or plausible soever, to be esteemed lawful on any other account, than its being coincident with, or not repugnant to, the laws promulgated by the voice of nature and reason. It must indeed be allowed, that the rational deduction of those laws is founded in the intrinsic tendency they have to promote the well-being of mankind, on condition they are universally and constantly observed. But though it afterwards comes to pass, that they accidentally fail of that end, or even promote the contrary, they are nevertheless binding, as hath been already proved. In short, that whole difficulty may be resolved by the following distinction. In framing the general laws of nature, it is granted, we must be entirely guided by the public good of mankind, but not in the ordinary moral actions of our lives. Such a rule, if universally observed, hath, from the nature of things, a necessary fitness to promote the general well-being of mankind: therefore it is a law of nature. This is good reasoning. But if we should say, such an action doth in this instance produce much good, and no harm to mankind; therefore it is lawful: this were wrong. The rule is framed with respect to the good of mankind; but our practice must be always shaped immediately by the rule. They who think the public good of a nation to be the sole measure of the obedience due to the civil power, seem not to have considered this distinction.

XXXII. If it be said that some negative precepts, e. g. thou shalt not kill, do admit of limitation, since otherwise it were unlawful for the magistrate, for a soldier in a battle, or a man in his own defence, to kill another: I answer, when a duty is expressed in too general terms, as in this instance, in order to a distinct declaration of it, either those terms may be changed for others of a more limited sense, as *kill* for *murder*, or else from the general proposition remaining in its full latitude, exceptions may be made of those precise cases, which, not agreeing with the notion of murder, are not prohibited by the law of nature. In the former case there is a limitation, but it is only of the signification of a single term too general and improper, by substituting another more proper and particular in its place. In the latter case there are exceptions, but then they are not from the law of nature, but from a more general proposition, which besides that law, includes somewhat more, which must be taken away in order to leave the law by itself clear and determinate. From neither of which concessions will it follow, that any negative law of nature is limited to those cases only where its particular application promotes the public good, or admits all other cases to be excepted from it, wherein its being actually observed produceth harm to the public. But of this I shall have occasion to say more in the sequel. I have now done with the first head, which was to show, that there is an absolute, unlimited passive obedience due to the supreme power, wherever placed in any nation; and come to inquire into the grounds and reasons of the contrary opinion: which was the second thing proposed.

XXXIII. One great principle, which the pleaders for resistance make the ground-work of their doctrine, is, that the law of self-preservation is prior to all other engagements, being the very first and fundamental law of nature. Hence, say they, subjects are obliged by nature, and it is their duty, to resist the cruel attempts of tyrants, however authorized by unjust and bloody laws, which are nothing else but the decrees of men, and consequently must give way to those of God or nature. But perhaps, if we narrowly examine this notion, it will not be found so just and clear as some men may imagine, or, indeed, as at first sight it seems to be. For we ought to distinguish between a twofold signification of the terms law of nature; which words do either denote a rule or precept for the direction of the voluntary actions of reasonable agents, and in that sense they imply a duty; or else they are used to signify any general rule, which we observe to obtain in the works of nature, independent of the wills of men; in which sense no duty is implied. And in this last acceptation, I grant it is a general law of nature, that in every animal there be implanted a desire of self-preservation, which though it is the earliest, the deepest, and most lasting of all, whether natural or acquired appetites, yet cannot with any propriety be termed a moral duty. But if in the former sense of the words, they mean that self-preservation is the first and most fundamental law of nature, which therefore must take place of all other natural or moral duties: I think that assertion to be manifestly false, for this plain reason, because it would thence follow, a man may lawfully commit any sin whatsoever to preserve his life, than which nothing can be more absurd.

XXXIV. It cannot indeed be denied, that the law of nature restrains us from doing those things which may injure the life of any man, and consequently our own. But, notwithstanding all that is said of the obligativeness and priority of the law of self-preservation, yet, for aught I can see, there is no particular law which obliges any man to prefer his own temporal good, not even life itself, to that of another man, much less to the observation of any one moral duty. This is what we are too ready to perform of our own accord; and there is more need of a law to curb and restrain, than there is of one to excite and inflame our self-love.

XXXV. But, secondly, though we should grant the duty of self-preservation to be the first and most necessary of all the positive or affirmative laws of nature; yet, forasmuch as it is a maxim allowed by all moralists, that evil is never to be committed, to the end good may come of it, it will thence plainly follow, that no negative precept ought to be transgressed for the sake of observing a positive one; and therefore, since we have shown 'thou shalt not resist the supreme power' to be a negative law of nature, it is a necessary consequence, that it may not be transgressed under pretence of fulfilling the positive duty of self-preservation.

XXXVI. A second erroneous ground of our adversaries, whereon they lay a main stress, is that they hold the public good of a particular nation to be the measure of the obedience due from the subject to the civil power, which therefore may be resisted whensoever the public good shall verily seem to require it. But this point hath been already considered, and in truth it can give small difficulty to whoever understands loyalty to be on the same foot with other moral duties enjoined in negative precepts, all which, though equally calculated to promote the general well-being, may not nevertheless be limited or suspended under pretext of giving way to the end, as is plain from what hath been premised on that subject.

XXXVII. A third reason which they insist on, is to this effect. All civil authority or right is derived originally from the people: but nobody can transfer that to another, which he hath not himself; therefore, since no man hath an absolute, unlimited right over his own life, the subject cannot transfer such a right to the prince (or supreme power), who consequently hath no such unlimited right to dispose of the lives of his subjects. In case therefore a subject resist his prince, who, acting according to law, maketh an unjust, though legal attempt on his life, he does him no wrong; since wrong it is not, to prevent another from seizing what he hath no right to: whence it should seem to follow, that agreeably to reason, the prince, or supreme power, wheresoever placed, may be resisted. Having thus endeavoured to state their argument in its clearest light, I make this answer. First, it is granted, no civil power hath an unlimited right to dispose of the life of any man. Secondly, in case one man resist another invading that which he hath no right to, it is granted he doth him no wrong. But in the third place, I deny that it doth thence follow, the supreme power may consonantly to reason be resisted; because that although such resistance wronged not the prince or supreme power wheresoever placed, yet it were injurious to the author of nature, and a violation of his law, which reason obligeth us to transgress upon no account whatsoever, as hath been demonstrated.

XXXVIII. A fourth mistake or prejudice which influenceth the impugners of non-resistance, arises from the natural dread of slavery, chains, and fetters, which inspires them with an aversion for any thing which even metaphorically comes under those denominations. Hence they cry out against us that we would deprive them of their natural freedom, that we are making chains for mankind, that we are for enslaving them, and the like. But how harsh soever the sentence may appear, yet it is most true, that our appetites, even the most natural, as of ease, plenty, or life itself, must be chained and fettered by the laws of nature and reason. This slavery, if they will call it so, or subjection of our passions to the immutable decrees of reason, though it may be galling to the sensual part, or the beast, yet sure I am, it addeth much to the dignity of that which is peculiarly human in our composition. This leads me to the fifth fundamental error.

XXXIX. Namely, the mistaking the object of passive obedience. We should consider, that when a subject endures the insolence and oppression of one or more magistrates, armed with the supreme civil power, the object of his submission is, in strict truth, nothing else but right reason, which is the voice of the author of nature. Think not we are so senseless, as to imagine tyrants cast in a better mould than other men: no, they are the worst and vilest of men, and for their own sakes have not the least right to our obedience. But the laws of God and nature must be obeyed, and our obedience to them is never more acceptable and sincere, than when it exposeth us to temporal calamities.

XL. A sixth false ground of persuasion to those we argue against, is their not distinguishing between the natures of positive and negative duties. For, say they, since our active obedience to the supreme civil power is acknowledged to be limited, why may not our duty of non-resistance be thought so too? The answer is plain; because positive and negative moral precepts are not of the same nature, the former admitting such limitations and exceptions as the latter are on no account liable to, as hath been already proved. It is very possible that a man in obeying the commands of his lawful governors, might transgress some law of God contrary to them; which it is not possible for him to do, merely by a patient suffering and non-resistance for conscience sake. And this furnishes such a satisfactory and obvious solution of the forementioned difficulty, that I am not a little surprised to see it insisted on, by men, otherwise, of good sense and reason. And so much for the grounds and reasons of the adversaries of non-resistance. I now proceed to the third and last thing proposed, namely, the consideration of the objections drawn from the pretended consequences of non-resistance.

XLI. First then it will be objected, that in consequence of that notion, we must believe that God hath, in several instances, laid the innocent part of mankind under an unavoidable necessity of enduring the greatest sufferings and hardships without any remedy; which is plainly inconsistent with the divine wisdom and goodness: and therefore the principle from whence that consequence flows, ought not to be admitted as a law of God or nature. In answer to which I observe, we must carefully distinguish between the necessary and accidental consequences of a moral law. The former kind are those which the law is in its own nature calculated to produce, and which have an inseparable connexion with the observation of it; and indeed if these are bad, we may justly conclude the law to be so too, and consequently not from God. But the accidental consequences of a law have no intrinsic natural connexion with, nor do they, strictly speaking, flow from its observation, but are the genuine result of something foreign and circumstantial, which happens to be joined with it. And these accidental consequences of a very good law may nevertheless be very bad; which badness of theirs is to be charged on their own proper and necessary cause, and not on the law, which hath no essential tendency to produce them. Now though it must granted, that a lawgiver infinitely wise and good will constitute such laws for the regulation of human actions, as have in their own nature a necessary inherent aptness to promote the common good of all mankind, and that in the greatest degree that the present circumstances and capacities of human nature will admit; yet we deny that the wisdom and goodness of the lawgiver are concerned, or may be called in question, on account of the particular evils which arise, necessarily and properly, from the transgression of some one or more good laws, and but accidentally from the observation of others. But it is plain that the several calamities and devastations, which oppressive governments bring on the world, are not the genuine, necessary effects of the law, that enjoineth a passive subjection to the supreme power, neither are they included in the primary intention thereof, but spring from avarice, ambition, cruelty, revenge, and the like inordinate affections and vices raging in the breasts of governors. They may not therefore argue a defect of wisdom or goodness in God's law, but of righteousness in men.

XLII. Such is the present state of things, so irregular are the wills, and so unrestrained the passions of men, that we every day see manifest breaches and violations of the laws of nature, which being always committed in favour of the wicked, must surely be sometimes attended with heavy disadvantages and miseries, on the part of those who by a firm adhesion to his laws endeavour to approve themselves in the eyes of their Creator. There are, in short, no rules of morality, not excepting the best, but what may subject good men to great sufferings and hardships, which necessarily follows from the wickedness of those they have to deal with, and but accidentally from those good rules. And as on the one hand it were inconsistent with the wisdom of God, by suffering a retaliation of fraud, perjury, or the like, on the head of offenders, to punish one transgression by another: so on the other hand, it were inconsistent with his justice, to leave the good and innocent a hopeless sacrifice to the wicked. God therefore hath appointed a day of retribution in another life, and in this we have his grace and a good conscience for our support. We should not therefore repine at the divine laws, or show a frowardness or impatience of those transient sufferings they accidentally expose us to, which, however grating to flesh and blood, will yet seem of small moment, if we compare the littleness and fleetingness of this present world with the glory and eternity of the next.

XLIII. From what hath been said I think it is plain, that the premised doctrine of non-resistance were safe, though the evils incurred thereby should be allowed never so great. But perhaps upon a strict examination they will be found much less than by many they are thought to be. The mischievous effects which are charged on that doctrine may be reduced to these two points. First, that it is an encouragement for all governors to become tyrants, by the prospect it gives them of impunity or non-resistance. Secondly, that it renders the oppression and cruelty of those who are tyrants, more insupportable and violent, by cutting off all opposition, and consequently all means of redress. I shall consider each of these distinctly. As to the first point, either you will suppose the governors to be good or ill men. If they are good, there is no fear of their becoming tyrants. And if they are ill men, that is, such as postpone the observation of God's laws to the satisfying of their own lusts, then it can be no security to them, that others will rigidly observe those moral precepts, which they find themselves so prone to transgress.

XLIV. It is indeed a breach of the law of nature for a subject, though under the greatest and most unjust sufferings, to lift up his hand against the supreme power. But it is a more heinous and inexcusable violation of it, for the persons invested with the supreme power, to use that power to the ruin and destruction of the people committed to their charge. What encouragement therefore can any man have to think that others will not be pushed on by the strong implanted appetite of self-preservation, to commit a crime, when he himself commits a more brutish and unnatural crime, perhaps without any provocation at all? Or is it to be imagined that they who daily break God's laws, for the sake of some little profit or transient pleasure, will not be tempted by the love of property, liberty, or life itself, to transgress that single precept which forbids resistance to the supreme power?

XLV. But it will be demanded, to what purpose then is this duty of non-resistance preached, and proved, and recommended to our practice, if in all likelihood, when things come to an extremity, men will never observe it? I answer, to the very same purpose that any other duty is preached. For what duty is there which many, too many, upon some consideration or other, may not be prevailed on to transgress? Moralists and divines do not preach the duties of nature and religion, with the view of gaining mankind to a perfect observation of them; that they know is not to be done. But, however, our pains are answered, if we can make men less sinners than otherwise they would be; if by opposing the force of duty to that of present interest and passion, we can get the better of some temptations, and balance others, while the greatest still remain invincible.

XLVI. But granting those who are invested with the supreme power to have all imaginable security, that no cruel and barbarous treatment whatever could provoke their subjects to rebellion: yet I believe it may be justly questioned, whether such security would tempt them to more or greater acts of cruelty, than jealousy, distrust, suspicion, and revenge may do in a state less secure. And so far in consideration of the first point, namely, that the doctrine of non-resistance is an encouragement for governors to become tyrants.

XLVII. The second mischievous effect it was charged with, is, that it renders the oppression and cruelty of those who are tyrants more insupportable and violent, by cutting off all opposition, and consequently all means of redress. But, if things are rightly considered, it will appear, that redressing the evils of government by force, is at best a very hazardous attempt, and what often puts the public in a worse state than it was before. For either you suppose the power of the rebels to be but small, and easily crushed, and then this is apt to inspire the governors with confidence and cruelty. Or, in case you suppose it more considerable, so as to be a match for the supreme power supported by the public treasure, forts, and armies, and that the whole nation is engaged in a civil war; the certain effects of this are rapine, bloodshed, misery, and confusion to all orders and parties of men, greater and more insupportable by far, than are known under any the most absolute and severe tyranny upon earth. And it may be that after much mutual slaughter, the rebellious party will prevail. And if they do prevail to destroy the government in being, it may be they will substitute a better in its place, or change it into better hands. And may not this come to pass without the expense, and toil, and blood of war? Is not the heart of a prince in the hand of God? may he not therefore give him a right sense of his duty, or may he not call him out of the world by sickness, accident, or the hand of some desperate ruffian, and send a better in his stead? When I speak as of a monarchy, I would be understood to mean all sorts of government, wheresoever the supreme power is lodged. Upon the whole, I think we may close with the heathen philosopher, who thought it the part of a wise man, never to attempt the change of government by force, when it could not be mended without the slaughter and banishment of his countrymen: but to sit still, and pray for better times.* For this way may do, and the other may not do; there is uncertainty in both courses. The difference is, that in the way of rebellion we are sure to increase the public calamities, for a time at least, though we are not sure of lessening them for the future.

XLVIII. But though it should be acknowledged, that in the main, submission and patience ought to be recommended: yet, men will be still apt to demand, whether extraordinary cases may not require extraordinary measures; and therefore in case the oppression be insupportable, and the prospect of deliverance sure, whether rebellion may not be allowed of? I answer, by no means. Perjury, or breach of faith, may, in some possible cases, bring great advantage to a nation, by freeing it from conditions inconsistent with its liberty and public welfare. So likewise may adultery, by procuring a domestic heir, prevent a kingdom's falling into the hands of a foreign power, which would in all probability prove its ruin. Yet will any man say, the extraordinary nature of those cases can take away the guilt of perjury and adultery? This is what I will not suppose.† But it hath been shown, that rebellion is as truly a crime against nature and reason as either of the foregoing, it may not therefore be justified upon any account whatever, any more than they.

* Plato in Epist. 7.

† When I wrote this, I could not think any man would avow the justifying those crimes on any pretext: but I since find that an author (supposed the same who published the book entitled, The Rights of the Christian Church), in a discourse concerning obedience to the supreme powers, printed with three other discourses at London, in the year 1709, ch. 4, p. 28, speaking of divine laws, is not ashamed to assert, "there is no law which wholly relates to man, but ceases to oblige, if upon the infinite variety of circumstances attending human affairs, it happens to be contrary to the good of man." So that, according to this writer, parricide, incest, or breach of faith, become innocent things, if, in the infinite variety of circumstances, they should happen to promote (or be thought by any private person to promote) the public good. After what has been already said, I hope I need not be at any pains to convince the reader of the absurdity and perniciousness of this notion. I shall only observe, that it appears the author was led into it by a more than ordinary aversion to passive obedience, which put him upon measuring or limiting that duty, and, with equal reason, all others, by the public good, to the entire unhinging of all order and morality among men. And it must be owned the transition was very natural.

XLIX. What! must we then submit our necks to the sword? and is there no help, no refuge against extreme tyranny established by law? In answer to this, I say in the first place, it is not to be feared that men in their wits should seek the destruction of their people, by such cruel and unnatural decrees as some are forward to suppose. I say, secondly, that in case they should, yet most certainly the subordinate magistrates may not, nay, they ought not, in obedience to those decrees, to act any thing contrary to the express laws of God. And perhaps, all things considered, it will be thought, that representing this limitation of their active obedience by the laws of God or nature, as a duty to the ministers of the supreme power, may prove in those extravagant supposed cases no less effectual for the peace and safety of a nation, than preaching up the power of resistance to the people.

L. Further it will probably be objected as an absurdity in the doctrine of passive obedience, that it enjoineth subjects a blind, implicit submission to the decrees of other men; which is unbecoming the dignity and freedom of reasonable agents; who indeed ought to pay obedience to their superiors, but it should be a rational obedience, such as arises from a knowledge of the equity of their laws, and the tendency they have to promote the public good. To which I answer, that it is not likely a government should suffer much for want of having its laws inspected and amended, by those who are not legally entitled to a share in the management of affairs of that nature. And it must be confessed, the bulk of mankind are by their circumstances and occupations so far unqualified to judge of such matters, that they must necessarily pay an implicit deference to some or other, and to whom so properly as to those invested with the supreme power?

LI. There is another objection against absolute submission which I should not have mentioned but that I find it insisted on by men of so great note as Grotius and Puffendorf,* who think our non-resistance should be measured by the intention of those who first framed the society. Now, say they, if we suppose the question put to them whether they meant to lay every subject under a necessity of choosing death, rather than in any case to resist the cruelty of his superiors, it cannot be imagined they would answer in the affirmative. For this were to put themselves in a worse condition than that which they endeavoured to avoid by entering into society. For although they were before obnoxious to the injuries of many, they had nevertheless the power of resisting them. But now they are bound, without any opposition at all, to endure the greatest injuries from those whom they have armed with their own strength. Which is by so much worse than the former state, as the undergoing an execution is

* Grotius de Jure Belli et Pacis, lib. i. c. iv. § 7, and Puffendorf de Jure Naturæ et Gentium, lib. vii. c. viii. § 7.

worse than the hazard of a battle. But (passing by all other exceptions which this method of arguing may be liable to) it is evident that a man had better be exposed to the absolute, irresistible decrees, even of one single person, whose own and posterity's true interest it is to preserve him in peace and plenty, and protect him from the injuries of all mankind beside, than remain an open prey to the rage and avarice of every wicked man upon earth, who either exceeds him in strength or takes him at an advantage. The truth of this is confirmed, as well by the constant experience of the far greater part of the world, as by what we have already observed concerning anarchy, and the inconsistence of such a state with that manner of life which human nature requires. Hence it is plain the objection last mentioned is built on a false supposition; viz. That men, by quitting the natural state of anarchy for that of absolute non-resisting obedience to government, would put themselves in a worse condition than they were in before.

LII. The last objection I shall take notice of is, that in pursuance of the premised doctrine, where no exceptions, no limitations are to be allowed of, it should seem to follow men were bound to submit without making any opposition to usurpers, or even madmen possessed of the supreme authority. Which is a notion so absurd and repugnant to common sense, that the foundation on which it is built may justly be called in question. Now in order to clear this point I observe the limitation of moral duties may be understood in a twofold sense, either first as a distinction applied to the terms of a proposition, whereby that which was expressed before too generally is limited to a particular acceptation: and this, in truth, is not so properly limiting the duty as defining it. Or, secondly, it may be understood as a suspending the observation of a duty for avoiding some extraordinary inconvenience, and thereby confining it to certain occasions. And in this last sense only, we have shown negative duties not to admit of limitation. Having premised this remark, I make the following answer to the objection. Namely, that by virtue of the duty of non-resistance, we are not obliged to submit the disposal of our lives and fortunes to the discretion either of madmen or of all those who by craft or violence invade the supreme power. Because the object of the submission enjoined subjects by the law of nature is, from the reason of the thing, manifestly limited so as to exclude both the one and the other. Which I shall not go about to prove, because I believe nobody has denied it. Nor doth the annexing such limits to the object of our obedience, at all limit the duty itself in the sense we except against.

LIII. In morality the eternal rules of action have the same immutable, universal truth with propositions in geometry. Neither of them depend on circumstances or accidents, being at all times and in all places, without limitation or exception, true. 'Thou shalt not resist the supreme civil power' is no less constant and unalterable a rule, for modelling the behaviour of a subject toward the government, than 'multiply the height by half the base,' is for measuring a triangle. And as it would not be thought to detract from the universality of this mathematical rule that it did not exactly measure a field which was not an exact triangle, so ought it not to be thought an argument against the universality of the rule prescribing passive obedience, that it does not reach a man's practice in all cases where a government is unhinged, or the supreme power disputed. There must be a triangle, and you must use your senses to know this, before there is room for applying your mathematical rule. And there must be a civil government, and you must know in whose hands it is lodged, before the moral precept takes place. But where the supreme power is ascertained, we should no more doubt of our submission to it than we would doubt of the way to measure a figure we know to be a triangle.

LIV. In the various changes and fluctuations of government it is impossible to prevent that controversies should sometimes arise concerning the seat of the supreme power. And in such cases subjects cannot be denied the liberty of judging for themselves, or of taking part with some and opposing others, according to the best of their judgments; all which is consistent with an exact observation of their duty, so long as, when the constitution is clear in the point, and the object of their submission undoubted, no pretext of interest, friends, or the public good, can make them depart from it. In short, it is acknowledged that the precept enjoining non-resistance is limited to particular objects, but not to particular occasions. And in this it is like all other moral negative duties, which, considered as general propositions, do admit of limitations and restrictions, in order to a distinct definition of the duty; but what is once known to be a duty of that sort, can never become otherwise by any good or ill effect, circumstance, or event whatsoever. And in truth if it were not so, if there were no general inflexible rules, but all negative as well as positive duties might be dispensed with, and warped to serve particular interests and occasions, there were an end of all morality.

LV. It is therefore evident, that as the observation of any other negative moral law is not to be limited to those instances only, where it may produce good effects; so neither is the observation of non-resistance limited in such sort, as that any man may lawfully transgress it, whensoever, in his judgment, the public good of his particular country shall require it. And it is with regard to this limitation by the effects, that I speak of non-resistance as an absolute, unconditioned, unlimited duty.

Which must inevitably be granted, unless one of these three things can be proved; either first, that non-resistance is no moral duty: or secondly, that other negative moral duties are limited by the effects: or lastly, that there is something peculiar in the nature of non-resistance, which necessarily subjects it to such a limitation, as no other negative moral duty can admit. The contrary to each of which points, if I mistake not, hath been clearly made out.

LVI. I have now briefly gone through the objections drawn from the consequences of non-resistance, which was the last general head I proposed to treat of. In handling this and the other points, I have endeavoured to be as full and clear, as the usual length of these discourses would permit, and throughout to consider the argument with the same indifference, as I should any other part of general knowledge, being verily persuaded that men as Christians are obliged to the practice of no one moral duty, which may not abide the severest test of reason.

# ARITHMETIC

## DEMONSTRATED WITHOUT EUCLID OR ALGEBRA.

TO WHICH ARE ADDED,

SOME THOUGHTS CONCERNING SURDS, THE ATMOSPHERIC TIDE, AND THE ALGEBRAIC GAME.

BY * * * * BACHELOR OF ARTS,

TRINITY COLLEGE, DUBLIN.

# DEDICATION.

To that very promising youth, WILLIAM PALLISER, the only son of the most reverend the ARCHBISHOP of CASHEL, endowed with genius, sagacity, and learning beyond his years, and born with every quality suited to afford some great light and increase to the sciences, this treatise on arithmetic is, as a small pledge of devoted attachment, offered and dedicated by

THE AUTHOR.

# PREFACE.

I PERCEIVE and regret, that most votaries of mathematical science are blindfolded on the very threshold. Inasmuch as the mode of learning mathematics, at least with us, first to apply to arithmetic, then geometry, then algebra; and as we read Tacquet's arithmetic, which no one can thoroughly understand without having some knowledge of algebra, it hence happens, that most students in mathematics, whilst they carefully and successfully master the demonstrations of theorems of inferior utility, leave untouched the principles and reasonings of arithmetical operations, though these last are of such efficacy and value, that they give the most important aid, not only to other branches of mathematics, but to the interests of men of all denominations. Wherefore if any one, after a mathematical course, turn his attention back to Tacquet's work, he will observe many things demonstrated in an obscure manner, so as not so much to enlighten as to force conviction on the mind, being environed with a repulsive array of porisms and theorems.

Nor has any one else, that I am aware of, demonstrated the rules of arithmetic without the aid of algebra. Thinking, then, that it would be of service to beginners if I should set forth my thoughts on these subjects, I now publish them, after they have almost all been kept by me for nearly three years. Now as I have not only given the rules for working questions, but also the demonstrations of those rules, drawn from the proper and genuine principles of arithmetic, some will perhaps be surprised that this treatise is of less size than the common works on arithmetic, though they contain merely the practice. The reason of this is that I have been very brief, both as regards precept and example, in explaining the "wherefore" of operations, on which writers on arithmetic are, in general, very tedious: and yet this brevity, as I hope, has not caused any obscurity. For although the blind require that a guide should lead them by the hand at every step, yet for one proceeding by the clear light of demonstration, it is sufficient to be furnished with even a slender clue. Wherefore I am anxious, that all votaries of mathematics should apply their minds to master the reasons and grounds of the rules of arithmetic.

This is not so difficult as some might suppose. The demonstrations here brought forward are, if I mistake not, easy at once and concise, nor are the principles drawn from any other quarter; nothing borrowed from algebra or Euclid is taken for granted here; I always prefer to prove an operation by obvious and familiar reasoning *à priori*, than to have recourse to an *argumentum ad absurdum*, by means of a tedious chain of consecutive demonstrations. I have endeavoured to derive the theory of square and cube roots from the nature itself of arithmetical involution, which, in my opinion, seems better suited to explain complicated extraction of roots, than what is generally applied to this purpose from the second book of Euclid, or from the analysis of algebraical powers. The common rule for the alligation of various things is demonstrated with difficulty and in particular instances. I have therefore substituted for it one of my own, which scarcely needs demonstration. I have rejected "the rule of false," as it is ineffectual and nearly useless.

I have copied no one; I have trespassed on the intellectual stores of none. For my original purpose was to deduce the rules of arithmetical operations from their principles for my own amusement and exercise, and so to employ my leisure hours. I could not on this occasion, without justly incurring the charge of ingratitude, omit mention of the name of the Rev. John Hall, doctor of divinity, Vice-Provost of this college, and the worthy professor of Hebrew. To that excellent man I acknowledge my obligations on many accounts, and not the least, that by his exhortations I was excited to the delightful study of mathematics.

I have now explained my aim: impartial judges will decide how far I have attained it. To their candid examination I cheerfully submit these first-fruits of my studies, little regarding what sciolists or the malignant may think.

# ARITHMETIC.

## PART I.

### CHAPTER I.

#### OF NOTATION AND THE STATEMENT OF NUMBERS.

There are nine numeral signs 1, 2, 3, 4, 5, 6, 7, 8, 9, employed with the cypher (0) for expressing unlimited classes of numbers. The whole of this contrivance depends on the value of these signs increasing in a tenfold proportion. The series of numbers rising in value according to that law is divided into members or periods for convenience of statement. The subjoined table will completely explain this,

*Series of Numeral Signs.*

| | | |
|---|---|---|
| Hundreds . . . . .<br>Tens . . . . . .<br>Units . . . . . . | 349 · | of Quintillions. |
| Hundreds . . . . . .<br>Tens . . . . . .<br>Units . . . . . . | 758 · | Quadrillions. |
| Hundreds . . . . .<br>Tens . . . . . . .<br>Units . . . . . . | 192 · | Trillions. |
| Hundreds . . . . . .<br>Tens . . . . . .<br>Units . . . . . . | 003 · | Billions. |
| Hundreds . . . . .<br>Tens . . . . . .<br>Units . . . . . . | 505 · | Millions. |
| Hundreds . . . .<br>Tens . . . . . .<br>Units . . . . . . | 739 · | Thousands. |
| Hundreds . . . . .<br>Tens . . . . . .<br>Units }<br>Unesimal } . . . | 047. | Integers. |
| Decimal . . . . . .<br>Centesimal . . . . . | 32 · | Parts. |
| Unesimal . . . . .<br>Decimal . . . . .<br>Centesimal . . . . . | 568 · | Thousandths. |
| Unesimal . . . . .<br>Decimal . . . . . .<br>Centesimal . . . . . | 918 · | Millionths. |
| Unesimal . . . . . .<br>Decimal . . . . .<br>Centesimal . . . . . | 300 · | Billionths. |
| Unesimal . . . . .<br>Decimal . . . . . .<br>Centesimal . . . . . | 052 · | Trillionths. |
| Unesimal . . . . . .<br>Decimal . . . . . .<br>Centesimal . . . . . | 704 | Quadrillionths. |

in which is exhibited a series of numeral signs, set forth by threes, the members or periods advancing in thousandfold proportion, and the places in tenfold proportion. For instance, the figure in the units place, and marked by a point placed under it, denotes seven individuals, integral, or considered at least as integral; the next number on the right hand, three tenth parts of that integer, and the number which immediately precedes it denotes four tens of the same integers, and in this tenfold proportion each place exceeds that following it, and is exceeded by that preceding it.

Still further, since by an infinite multiplication and division of units, the series of signs is infinitely extended in each direction, from the units place, and so innumerable places for expressing their exact value, there is need merely of the continual repetition of three numbers, provided that each collection of threes, or period, be designated by its own name, as is the case in the table; for, in proceeding from the units place, towards the left, the first period marks units, or integers, the second thousands, the third millions, the fourth billions, and so on. In the same way, preserving the analogy, in the periods descending below units, first occur the parts simply, then thousandths, then millionths, then billionths, and so on; and these last are to be divided into unesimal or unit parts, tenths, hundredths, the others to be collected into units, tens, hundreds.

If then we wish to state the number expressed by any figure of the series, we must, 1st, note the simple value of the figure; 2nd, the value of the place; lastly, of the period. For instance, let the number selected be 9, in the fifth period towards the left. The figure, taken simply, has the value of nine; in consequence of its place, it has the value of nine tens; and in consequence of its period, of nine tens of trillions. Let 5 be chosen in the third period: taken simply, it signifies five; in consequence of its place, five units; in consequence of its period, five units of millions, or five millions. In the second period below unit, let 8 be chosen: the simple value of the figure is eight: in consequence of its place, eight hundredths; in consequence of its period, eight hundredths of thousandths.

If the number to be stated have not words affixed, to denote the value of the periods and of the places, it should be pointed into threes towards the right and left, from the units place, and then should be expressed by the name assigned to the place and period. For instance, let the numbers proposed be 73,480,195. The figures being divided into periods, I first inquire what is the value of the figure in the first place on the the left, which, since it is in the second place of the the third period, is seven tens of millions; but since the numbers advance in tenfold proportion, the value of the first figure being known, the values of the rest follow in due order. We shall therefore thus express the proposed number: seven tens, and three units of millions; four hundreds, and eight tens of thousands; one hundred, nine tens, and five units: or more concisely, seventy-three millions, four hundred and eighty thousand, a hundred and ninety-five. Hence we perceive that a cypher, though in itself of no value, must of necessity be expressed, for the purpose of assigning a proper place to each figure.

There will be no difficulty in writing and expressing the largest numbers, if due attention be given to what has been just laid down, an acquaintance with which will also afterwards be of the greatest importance; for nature itself teaches us the way of working arithmetical questions on the fingers, but there is need of science to perform these operations accurately, with respect to greater numbers; all turning upon this, that whereas the limited nature of our faculties does not permit the work to be done at once, and with a single effort, we divide it into various operations, by separately inquiring the aggregate or sum, the difference, the product, and then combining them, express the ultimate sum total, remainder, or product; the whole reason and contrivance of the operations resulting from the simple progression of the places, and being ultimately founded on it.

N.B. I am aware that some arithmeticians divide the series otherwise than I do; for compounding the denominations, they use sixes instead of threes. But as others* follow the method pursued by me, I have thought it advisable to retain it as simpler.

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## CHAP. II.

### ON ADDITION.

In addition, the sum of two or more numbers is required; to obtain which the numbers to be added should be set down, so that units should be placed under units, tens under tens, and decimal parts under decimals, and so forth. On this account, when decimal parts are added, the units place should be marked by the insertion of a comma. Then commencing from the right, the figures in the first place should be added, and if any tens result, they should be carried over to the next place, and be added to the sum of the figures of that place, the tens which belong to the next place being reserved, and so the process should be continued. For instance, in the first example of the operations below, 9 and 5 make 14; I therefore reserve the 10 and proceed with the 4; 4 and 8 make 12, therefore I reserve the 10,

* For instance, the celebrated Wallis in his Universal Mathesis, and Father Lamy, in his Elements of Mathematics.

and going on to the next place I find 6, to which I add 2, on account of the tens reserved in the first place; and as 8 and 2 make 10, I reserve that, and set down the 1 which remains, and proceed in the same way.

| | | | |
|---|---|---|---|
| Addend. | 2 0 1 8·<br>8·2 2·5·<br>4 3 6 9 | 523,9702<br>81,35<br>60,2005 | £ s. d.<br>7 8 9<br>3 12 5<br>0 7 2 |
| Sum | 1 4 6 1 2 | 665,5207 | 11 8 4 |

If the things to be summed up be of different kinds, we should proceed in the same way, taking into account, however, the proportions according to which the different denominations advance. For instance, the denominations of pounds, shillings, and pence, do not advance as those of numbers; for 12 pence, not 10 pence, make a shilling; 20 shillings, and not 10 shillings, make a pound. On this account, in adding such quantities, instead of tens, twelves should be carried from the pence, twenties should be carried from the shillings to the next place.

## CHAP. III.

### ON SUBTRACTION.

In subtraction, the difference of two numbers is required, or what remains after one has been taken from the other; for ascertaining which the less quantity of each denomination should be placed under the greater; then beginning from the right, the first denomination of the quantity to be subtracted is to be taken from that written above it, and the remainder set down below, and the work continued in this way until the whole subtraction be effected.

If, however, it should happen that any number be too small to admit of the lower quantity being taken from it, such upper number should be increased by ten, that is, by a unit borrowed from the next place.

Let it be required to subtract 1189 from 32034; the numbers being set down as in the adjoined example, I set about subtracting the first figure 9 from the 4 placed over it; but as 4 does not even once contain 9, a ten must be added to it, so as to make 14, and then 9 taken from 14 leaves 5. Proceeding then to the left, I have to subtract 8 from 2, not from 3, because we should take into account the ten which has been borrowed, and as

8 cannot be subtracted from 2, I subtract it from 12, and 4 remains. The next figure of the quantity to be subtracted is 1, which, however, cannot be subtracted from nothing or 0; in place of the cipher 0 I use 9; now I use 9, because the ten which is borrowed must be diminished by the unit which has been added to the preceding figure; continuing the process in this manner, 1 taken from 1 leaves nothing. Finally, the subtraction being completed, 3 remain, which I set down below.

In a similar manner, the subtraction of different denominations is effected: only we should observe that ten is not necessarily to be used, but such a number as declares how many of the denomination in question are in the next denomination, and this number should be borrowed to supply the defect of any particular figure.

| | | | £ s. d. |
|---|---|---|---|
| Subtract. | 32034<br>1189 | 7329,645<br>3042,100 | 4 8 3<br>2 6 5 |
| Rem. | 30845 | 4287,545 | 2 1 10 |

N.B. From what has been here laid down, it is plain that the science of arithmetic, as far as we have treated it, consists in doing in detail that which cannot be done at once; and that the reason of reserving tens in addition, and of borrowing them in subtraction, altogether depends on the tenfold advance in the value of the places.

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## CHAP. IV.

### ON MULTIPLICATION.

In multiplication the multiplicand is taken as often as the multiplier requires; or in other words, a number is sought bearing the same ratio to the multiplicand that the multiplier does to unity. That number is called the product, or rectangle; the factors or sides of which are called respectively, the multiplicand, and the number by which it is multiplied.

For finding the product of two numbers, the multiplying number being written under the multiplicand, this last should be multiplied by each figure of the former, beginning from the right hand: the first figure of the product should be written directly under the multiplying figure, and the rest in order towards the left.

The multiplication being finished, the several products should be collected into one sum, the number of decimal places in which should be equal to those in both the factors.

Let 30,94 be the number to be multiplied by 26,5. Five times 4 produce 20, the first figure of which (0) I place under the multiplying figure (5), and carry the remaining 2; then 5 multiplied into 9 produces 45; 5 with the 2 carried make 7, which I set down, carrying 4 to the next place, and so on.

| | | | |
|---|---|---|---|
| | 30,94<br>26,5 | 52886<br>24 | 6000<br>56 |
| | 15470<br>18564<br>6188 | 211544<br>105772 | 36<br>30 |
| Prod. tot. | 819,910 | 1269264 | 336000 |

As there is a twofold value of each number, this should be taken into account, so that the multiplication be rightly effected, that is, that each figure be multiplied as well according to the simple value of the multiplying figure, as according to that which it has from its place. Hence the figure of each respective product is written under the multiplying figure. For instance, in the multiplier of the second example, the figure 2 has the value, not of 2 units, but of 2 tens; therefore, when multiplied into 6, the first figure of the multiplicand, it will produce, not 12 units, but 12 tens; therefore, the first figure of the product should be set down in the place of tens, that is, directly under the multiplying figure 2.

For the same reason, when there are (decimal) parts in the factors, the number produced by the multiplication of the first figure of the multiplier into the first of the multiplicand, is to be removed as far below the multiplied figure as the multiplier is to the right hand below unity; so that as many (decimal) places in the entire product are to be marked off as there were in both factors.

N.B. If there be ciphers continuously to the right of each or of both factors, the multiplication should be performed on the other figures merely, and the ciphers afterwards annexed to the entire product; for since the places advance in value in a tenfold proportion, it is clear that a number becomes tenfold, a hundredfold, a thousandfold itself, if it be advanced one, two, or three places.

# CHAP. V.

## ON DIVISION.

DIVISION is the reverse of multiplication, its object being to resolve or divide that quantity which the latter produces. The number found by division is called the quotient, because it declares how often the dividend contains the divisor, or, what is the same, the ratio of the dividend to the divisor, or finally, the part of the dividend denominated from the divisor.

In division, having written down the dividend and divisor, as in the first of the subjoined examples, commencing from the left, that part of the dividend containing the divisor, or having the least excess above the containing number, should be marked off by a point. I mean in this instance the simple values. It should then be ascertained how often the divisor is contained in that member of the dividend, and the resulting number will be the first figure of the quotient; the divisor should then be multiplied into the figure thus found, and the product subtracted from the member of the dividend, and the remainder set down below; to which should be annexed the next figure of the dividend, and a new dividend thus obtained, from whence must be ascertained the next figure of the quotient, which being multiplied into the divisor, and the product subtracted from the dividend just divided, the remainder with the next figure of the original dividend annexed will form a new member, and so on, until the operation is finished. The decimal places of the divisor being then subtracted from those in the dividend, the remainder will indicate what number of places should be in the quotient; but if this subtraction be not feasible, so many decimal cyphers should be added to the dividend as are necessary.

If, after the division has been completed, there should be a remainder, by adding decimal cyphers the division can be continued until either nothing remain, or it be so minute that it need not be taken into account, or the remaining figures may be set down and the divisor under them.

If both the dividend and divisor end in cyphers, an equal number of these should be struck off in both: but if the divisor alone end in cyphers they should not be taken into account in the operation, but the same number of the last figures of the dividend should be struck off, and at the end of the work set down, a line drawn below them, and the divisor written underneath.

Let it be required to divide 45832 by 67. Since the divisor is greater than 45 let another figure be added, and the member taken for division be 458, which I separate from the rest of the dividend by a point. 6 is contained in 45 seven times, with three remainder; but, as 7 is not also contained 7 times in 38, the quotient must be taken less. Let 6 therefore be taken; and, as 6 is contained 6 times in 45 and 9 remains, and 98 contains 7 six times, the first figure of the quotient should be 6. This, multiplied into the divisor, produces a subtrahend 402, which, being taken from 458, there will be a remainder 56; to which I annex 3, the next figure of the dividend, by which means a new dividend is formed 563, which I divide as the former, and find 8 for the second figure of the quotient; and, as 8 multiplied into 67 produces 536, I subtract this from 563, and, adding to the remainder 2, the next figure of the dividend, I obtain 272 as a new dividend, which, when divided, gives 4, which, being first set down in the quotient, and then multiplied into the divisor, and the product subtracted from 272, there remains 4, which should be annexed to the quotient, a line being drawn under it, and the divisor then written below it.

The operation is more speedy when the subtraction immediately follows the multiplication of each figure, and the multiplication proceeds from left to right. For instance, let it be proposed to divide 12199980 by 156, as in the third example; the divisor being written under 1219, the first member of the dividend, it is plain that the one is contained 7 times in the other; and, in consequence, 7 is put down in the quotient. Seven times 1 make 7; which, being subtracted from 12, I strike out both the multiplied figure 1 and 12, the part of the member from which the product was subtracted, setting down above the remainder 5; then I proceed to 5, the next figure of the divisor; 7 multiplied into 5 makes 35, and 35 being subtracted from 51, there remains 16, which I write above and strike out 51 and 5. I then multiply 7 into 6, and the product 42 being subtracted from 69 there remain 27, which I set down, striking out both 69 and 6, the last figure of the dividend. The divisor being now entirely struck out, I set it down, moved one place to the right, and with it I divide the member written above it, which indeed is made up of the remainder of the last divided member increased by the following figure. In this way the divisor should be moved until it goes through the whole dividend.

| | | |
|---|---|---|
| 67)458.32(684$\frac{4}{67}$<br>402<br>——<br>563<br>536<br>——<br>272<br>268<br>——<br>004 | 200)8200<br><br>2)82(41<br>8<br>—<br>02<br>2<br>—<br>00 | 41<br>123 2<br>5671 73<br>12199980(78205<br>1566666<br>15555<br>111 |

In the next place the reason of the rules will be given; and first, it is plain why we should seek for the quotient by separate divisions.

2ndly, It may, for instance, be asked why, in the example above given, 6 should be taken as the quotient of the first member, divided by the divisor; for 67 is contained in 458 hundreds, not six times but six hundred times, for they are not units but hundreds, since they are distant two places to the left from units.

To this I answer, that in reality not merely 6 but 600 is written in the quotient, for two figures afterwards ascertained follow it, and thus the proper value is assigned to the quotient; for as many places are set down after each figure in the quotient as there are after that member of the dividend from which they are obtained.

3rdly, Since each figure of the quotient indicates how often that member of the dividend from which it was obtained contains the divisor, it is proper that the subtrahend should be formed by multiplying the divisor into the figure last found; for then the divisor is subtracted exactly as often as it is contained in the dividend, unless it should happen that the number last set down in the quotient should be too large or too small: if the first be the case the product will be so large that it cannot be subtracted; if the latter be the case, then the resulting product will be so small, that after performing the subtraction the remainder will be equal to or greater than the divisor.

4thly, The reason why so many (decimal) places should be marked off in the quotient that with those which are in the divisor they may be equal to those in the dividend, is that the dividend is the product resulting from the multiplication of the divisor into the quotient, and consequently it should have as many decimal places as those two, as we have shown when treating of multiplication.

5thly, It is clear that decimal cyphers, annexed to the end of the dividend, do not affect its value; for as to the integers, those which are removed for the same distance from the units' place, have the same value, but decimals are not diminished in value unless the cyphers be placed before them.

6thly, Since the quotient expresses or denominates the ratio of the dividend to the divisor, it is clear, that as long as that continues the same the quotient must be the same; but the common cyphers, being cast away, the ratio of the numbers to each other is not in the slightest degree altered. Thus, for instance, 200 bears the same ratio to 100—that is, 200 contains 100 as often as 2 contains 1, which is sufficiently manifest by itself.

# CHAP. VI.

## ON FORMING SQUARES AND FINDING THEIR ROOTS.

THE product of a number multiplied into itself is called its square; and the number by the multiplication of which into itself the square is produced is called the square root or side; and the operation by which we search for the root of the given square is called the extraction of the square root; for understanding which it will be necessary to consider the manner in which the square is produced, the parts of which it is composed, their order, and relative position. And since it is best in acquiring a knowledge of any thing to proceed from the simplest and easiest, let us commence with the consideration of the production of a square, resulting from a binomial root.

We should, in the first place, closely consider what takes place when a number, consisting of two figures, is multiplied into itself. And first it is plain that the first figure on the right of the root is multiplied into the one placed above it, that is, into itself, and thence results the square of the lesser number. Then, by multiplying the same figure into the next part of the multiplicand, a rectangle results contained by both members of the root. Having then finished the multiplication of the whole multiplicand by the first figure of the root, we come to the second, which, being multiplied into the first figure of the multiplicand, there results again a rectangle, contained under the two figures of the binomial root; then the second figure of the multiplicand multiplied into itself, gives the square of the second member of the binomial root. We ascertain from this that any square produced from a binomial root consists, in the first place, of the square of the lesser member; in the second, of double the rectangle contained under the members; in the third, of the square of the greater member.

Let it be required to square a binomial root, for instance, 23, according to what has been laid down in Chap. IV. I first multiply 3 into 3, which gives 9, as the square of the first member; I secondly multiply 3 into 2, the other figure of the root, and obtain 6, the rectangle contained by both; thirdly, from 2 multiplied into 3 arises a second time, the rectangle contained under the members; in the fourth place, 2 multiplied into 2 produces 4 the square of the greater member.

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 23
 23
 --
 69
46
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Let us now proceed to the production of a square, from a root of three members. In this operation, the first figure of the root multiplied into the whole root, produces, in the first place, the square of the first member; in the second place, the rectangle contained under the first and second member; in the third place, the rectangle contained under the first and third member. Now for the second figure, this multiplied into the root gives first the rectangle contained under the first and second member; secondly, the square of the second member; thirdly, the rectangle contained under the second and third members. Lastly, from the third figure of the root, multiplied into the root, results first, the rectangle contained under the first and third members; secondly, the rectangle contained under the second and third members; and thirdly, the square of the third member.

From this we ascertain, that a square, produced from a trinomial root comprises, first, the square of the first figure of the root; secondly, double the rectangle contained under the first figure and the two others; in the third place, the square of the two others, that is, the square of each, and also double the rectangle contained under both, which we have before shown, constitute the square of the two figures.

In the same way, it can be shown, that the square of four, five, or any number of figures, contains, first, the square of the figure of lowest value; secondly, double the rectangle resulting from the multiplication of the figure of lowest value into all the others; thirdly, the square of all the other figures, which itself, as is plain from what has been stated, contains the square of the figure next from the right, double the rectangle of that same figure multiplied into all the others, the square of all the other figures, which in the same way contains the square of the third figure, two rectangles of it, and the others, and the square of these, and so on, until we come to the square of the highest figure of the root.

The parts of which the square is composed being ascertained, we should next consider, concerning their arrangement and place. If therefore, beginning from the right, we divide the square into periods of twos, from the mode of production which we have explained above, it is plain that the first member from the left will be occupied by the square of the first or highest figure, and at the same time, of that portion of double the rectangle resulting from the multiplication together of the first and second figures, which is redundant above the first place of the following period of two; that the first place of the second period contains double the rectangle mentioned, and besides whatever of the square of the second figure is over; that the second contains the square of the second figure, and whatever is over of double the rectangle of the two first figures, multiplied into the third as far as the lowest figure at the first place of the third two, and so on.

For instance, in the annexed example, the first member 10 contains 9, the square of the first figure 3, besides 1 by which 12 (double the rectangle of the figure 3 multiplied into the following 2) exceeds the first place of the second member. The first place of the second two contains 2 (the remainder of the double rectangle of the figures 3 and 2), and also that which is over the next following place, and so on, &c.

```
      321
      321
   ------
      321
     642
    963
  -------
  10.30.41
```

Having thus considered the formation of the square, let us proceed to its analysis. Let any number, for instance 10.30.41, be proposed, the square root of which is required. This should be pointed off by twos, beginning from the right in case the number be even, as if otherwise the last member will consist of but one figure. I then inquire, what is the greatest square contained in 10, the first member towards the left, and 3, the root of this, is the first figure of the root required, the square of which (9) I subtract from (10) the member. From the remainder (1), with (3) the first figure of the following member added, is formed a dividend (13), which I divide by the found figure doubled (6), the quotient (2) will be the second figure of the root; which being multiplied first into the divisor, and then into itself, and the sum of the product, taken so, however, that the latter be removed one place to the right (124), I take away this from the dividend (13), increased by 0, the remaining figure of the second member. To the remainder 6, I add 4, the first figure of the third two, and so a new dividend (64) is produced, which being divided by 64, twice the root already found, gives 1, the third of the required root; this being then multiplied into itself and the products added up, I subtract the sum (641) from the dividend, increased by the addition of the other figure of the third member, and in this way we must proceed to whatever length the operation may be carried.

If after the last subtraction there be a remainder, it shows that the given number was not a square; however by adding to it decimal cyphers, the operation can be continued to any extent thought desirable.

If there be any decimal places in the number, for the root of which we are searching, their number divided by two, will show how many should be in the root. The reason of this appears from Chap. IV.

The reason of the mode of proceeding is quite clear from what has been stated. For as a divisor I employed 6, the double of the found figure, because, from the formation of the square as it has been explained, I knew that double the rectangle of that figure, multiplied into the following one, comprised the dividend; consequently, if this were divided by the double of one factor (3), that the other factor (2), that is, the next figure of the root, could be obtained. So likewise I have formed a subtrahend from double the rectangle of the quotient and divisor, and the square of the quotient added together, because I found that those two rectangles and the square were contained in that order in the remainder and the following member from which the subtraction was made, and so the evolution of the power is easily effected from its involution or formation.

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## CHAP. VII.

### CONCERNING THE INVOLUTION AND EVOLUTION OF THE CUBE.

THE root multiplied into the square produces the cube. To prepare the way for the analysis we should, as has been done in the former chapter, begin with the composition of the power. In the production then of the cube from a binomial root, the first member of the root, in the first place, meets with its own square, whence results the cube of the first figure; secondly, double the rectangle of the members, whence double the solid of the square of the first figure multiplied into the other; thirdly, the square of the other member, whence the solid produced from the first figure and the square of the second. In the same way, when the multiplication takes place by the second member, there arises the solid of the second figure and of the square of the first; in the second place double the solid of the first figure and of the square of the second; in the third place the cube of the second member.

Therefore the cube produced from the binomial root contains the cubes of the two members and six solids, that is to say, three made from the square of each member, multiplied into the other.

The reasoning being continued according to the analogy of the preceding chapter, it will follow, that if, as the square should be divided into twos, the cube resulting from any root be distributed into threes, that the three, or member first from the left, contains the cube of the figure first on the left, and also the excess, if there be any, of three solids of the square of the same, multiplied into the second; that the first place of the second contains the said solids and the excess of the three solids of the square of the second figure, multiplied into the first; that the second place contains the same three solids and the excess of the cube of the second figure; and that the third is occupied by the said cube and the excess of the three solids produced from the square of the preceding figures, multiplied into the third; and that the solids just mentioned fill the first place of the third member, and so on. From this we shall easily derive the following manner of extracting the cube root.

Beginning from the right, I divide, by means of points, the resolvend (80621568) into threes, except the last member, which can be less. I then take the greatest cube (64) contained in the first member towards the left, and having written down its root (4) for the first figure of the sought root, I annex to the remainder (16) the next figure (6) of the resolvend, whence results a dividend (166), which I divide by 48, thrice the square of the figure which has been found: the quotient (3) is the second figure of the root. I multiply this first into the divisor, secondly its square into three times the first figure, and thirdly itself into itself twice. The products then being collected in this way, that the second be set down one place to the right of the first, the third one place to the right of the second. $\left\{\begin{array}{r}144\phantom{00}\\108\phantom{0}\\27\end{array}\right\}$ I subtract it from the dividend increased by the addition of the two remaining figures of the second member. In this way, however prolonged the operation, a dividend will always result from the remainder, with the addition of the first figure of the following member, and a divisor from three times the square of the figures of the root already found, and a subducend from the figure last found, the square of the same multiplied into three times the preceding figures, lastly its cube, and these collected in the manner set forth.

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     80.621.568(432
     64
    ------
  48)16621
     15507
    ------------
5547)1114568
     1114568
     -------
     0000000
```

If the resolvend be not a cube, by adding decimals to the remainder you can carry its exhaustion to infinity.

The root should have a third part of the decimal places of the resolvend.

N. B. Synthetical operations can be examined by means of analytical, and analytical by means of synthetical; so if either number, being subtracted from the sums of two numbers, the other remains, the addition has been rightly performed; and *vice versâ*, subtraction is proved to be right when the sum of the subtrahend and remainder is equal to the greater number. So if the quotient multiplied into the divisor produce the dividend, or the root multiplied into itself produce the resolvend, it is a proof that the division or evolution has been correct.

# ARITHMETIC.

## PART II.

### CHAP. I.

#### ON FRACTIONS.

It has been before mentioned that division is signified by setting down the dividend with the divisor under it, and separated from it by a line drawn between them. Quotients of this kind are called broken numbers, or fractions, because the upper number, called also the numerator, is divided or broken into parts, the denomination of which is fixed by the lower, which is therefore called the denominator. For instance in the fraction $\frac{2}{4}$, 2 is the dividend or numerator, 4 the divisor or denominator, and the fraction indicates the quotient which arises from 2 divided by 4, that is the fourth of any two things whatever, or two fourths of one, for they mean the same.

N. B. It is clear that numbers which denote decimal parts, and which are commonly called decimal fractions, can be expressed as vulgar fractions, if the denominator be written beneath. For instance, .25 is equivalent to $\frac{25}{100}$, .004 is equivalent to $\frac{4}{1000}$, &c., which we must either do, or understand to be done, as often as those are to be reduced to vulgar fractions, or conversely these are to be reduced into those, or any other operation is to take place equally affecting both fractions, decimal and vulgar.

### CHAP. II.

#### OF ADDITION AND SUBTRACTION OF FRACTIONS.

1. If fractions, whose sum or difference is sought, have the same denominator, the sum or difference of the numerators should be taken, and the common denominator written under, and this will be the answer.

2. If they be not of the same denomination let them be reduced to the same denomination. The denominators multiplied into each other will give a new denominator, but the numerator of each fraction multiplied into the denominators of the others will give a numerator of a new fraction of equal value. Then the new fractions should be treated as above.

3. If an integer is to be added to a fraction or subtracted from it, or *vice versâ*, it should be reduced to a fraction of the same denomination as the given one; that is, it is to be multiplied into the given denominator, and that denominator to be placed under it.

| | |
|---|---|
| Addition. | $\frac{1}{5}$ to $\frac{2}{5}$ sum $\frac{3}{5}$ |
| Subtraction. | $\frac{1}{3}$ from $\frac{2}{3}$ rem. $\frac{1}{3}$ |
| Addition. | $\frac{2}{3}$ to $\frac{3}{4}$, that is $\frac{8}{12}$ to $\frac{9}{12}$, sum $\frac{17}{12}$ |
| Subtraction. | $\frac{2}{3}$ from $\frac{3}{4}$, that is $\frac{8}{12}$ from $\frac{9}{12}$, rem. $\frac{1}{12}$ |
| Addition. | 3 to $\frac{5}{8}$, that is $\frac{24}{8}$ to $\frac{5}{8}$, sum $\frac{29}{8}$ |
| Subtraction. | $\frac{5}{8}$ from 3, that is $\frac{24}{8}$, rem. $\frac{19}{8}$ |

In the first place, it should be explained why fractions should be reduced to the same denomination before we treat them; and it is on this account, that numbers enumerating heterogeneous things cannot be added together, or subtracted from each other. For instance, if I wish to add three pence to two shillings, the sum will not be 5 shillings, or 5 pence, nor can it be ascertained before that the things mentioned be brought to the same sort, by using 24 pence instead of 2 shillings, to which if I add 3 pence, there results a sum of 27 pence; for the same reason, if I have to add 2 thirds and 3 fourths, I do not write down 5 parts either thirds or fourths, but, instead of them I employ 8 twelfths and 9 twelfths, the sum of which is 17 twelfths.

Secondly. I wish to show that fractions after such reduction are of the same value as before, for instance, $\frac{2}{3}$ and $\frac{8}{12}$; since both numerator and denominator are multiplied by the same number (4); but every fraction represents the ratio of the numerator or dividend to the denominator, or divisor, and consequently as long as that remains the same, the fraction retains the same value, but each term of the ratio being multiplied by the same number, it is certain that the ratio is not changed: for instance, if the half of one thing be double the half of another, that whole will be double this whole, which is so plain that it does not require proof.

Thirdly. An integer reduced to a fraction is not altered in value, for if the rectangle of two numbers be divided by one of them, the other will be quotient; but, in the reduction of an integer to a fraction, it is multiplied into the given denominator, and also divided by it, therefore the fraction has the same value as the given integer.

N. B. It will sometimes be useful to reduce a fraction to a given denominator, for instance, $\frac{2}{3}$ to another whose denominator is 9, which is done by means of the rule of three (laid down subsequently), by finding a number to which the given denominator will be as the denominator of the given fraction to its numerator; that will be the numerator of the fraction of which the name has been given, and the value will be the same as of the former, for, in each instance the ratio between the terms of the fraction is the same.

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## CHAP. III.

### OF THE MULTIPLICATION OF FRACTIONS.

1. If a fraction is to be multiplied into a fraction, the numerators of the given fractions multiplied into each other will give the numerator of the product, and the denominators in the same way will give the denominators.

2. If a fraction is to be multiplied into an integer, the given integer should be multiplied into the numerator of the fraction, the denominator remaining the same.

3. If in either factor, or in both, integers occur, or heterogeneous fractions, they, for the sake of clearness, should be collected together.

*Examples of Multiplication.*

| | |
|---|---|
| Multiply | $\frac{2}{3}$ by $\frac{5}{8}$, prod. $\frac{10}{24}$, $\frac{4}{7}$ by 2, prod. $\frac{8}{7}$. |
| Multiply | 2 and $\frac{3}{5}$ by $\frac{1}{2}$ and $\frac{2}{3}$, that is, $\frac{13}{5}$ by $\frac{7}{6}$. |

1. It is plain that the quotient is increased in the same proportion as is the dividend: for instance, if 2 be contained three times in 6 it will be contained twice three times in twice 6. It is plain also that it is diminished in the same proportion as the divisor increases: for instance, if the number 3 be contained 4 times in 12, twice 3 will be contained only twice in 12; therefore when I multiply $\frac{2}{3}$ by $\frac{5}{8}$, the fraction $\frac{2}{3}$ is to be increased in a fivefold ratio, since it is to be multiplied by 5; and to be diminished in an eightfold ratio, since it is multiplied, not actually by 5, but only by its eighth part; consequently I multiply the dividend 2 by 5, and the divisor 3 by 8.

2. As to the second rule, it is plain that twice 4 of any things are equal to 8 things of the same denomination, whatever it may be.

## CHAP. IV.

### ON DIVISION OF FRACTIONS.

1. A FRACTION is divided by an integer, by multiplying the given integer into the denominator of the given fraction.

2. If a fraction is to be divided by a fraction, the numerator of the divisor multiplied into the denominator of the dividend will give the denominator of the quotient; and its denominator multiplied into the numerator of the dividend will give the numerator of the quotient.

3. Whenever integers or fractions of different denominations are mixed, the easiest way of proceeding will be to collect the members of each, as well dividend as divisor, into two sums.

*Examples of Division.*

| |
|---|
| Divide $\frac{3}{4}$ by 2, quot. $\frac{3}{8}$. |
| Divide $\frac{4}{9}$ by $\frac{2}{5}$, quot. $\frac{20}{18}$. |
| Divide $2\frac{1}{3}\ \frac{1}{2}$ by $3\frac{2}{3}$, that is, $\frac{21}{6}$ by $\frac{17}{6}$. |

1. As to the first rule, it is clear from the preceding chapter that a fraction is lessened or divided in the same proportion as the denominator is multiplied.

2. Since to divide one fraction by another, for instance $\frac{4}{9}$ by $\frac{2}{5}$, I have multiplied the denominator 9 into 2, the fraction $\frac{4}{18}$ only expresses how often 2 is contained in the dividend; but the fifth of it will indicate how often the fifth part of the number 2 is contained in it; wherefore I multiply the first quotient $\frac{4}{18}$ by 5, whence results $\frac{20}{18}$.

N.B. If the given fractions be homogeneous, the shorter and more elegant way is to divide the numerator of the dividend by the numerator of the divisor as often as it measures it. Thus, $\frac{6}{8}$ being divided by $\frac{3}{8}$, the quotient will be 2, for whatever things are enumerated by 6 contain 3 twice.

2. If a root is to be extracted from a given fraction, the root of the denominator, placed under the root of the numerator, will form a fraction, which is the root sought. For instance $\frac{2}{3}$ is the square root of the fraction $\frac{4}{9}$, and the cube root of the fraction $\frac{8}{27}$; for, from what we have said about multiplication it is clear that $\frac{2}{3}$ multiplied into $\frac{2}{3}$ produces $\frac{4}{9}$; and that $\frac{2}{3}$, multiplied into $\frac{4}{9}$, produces $\frac{8}{27}$.

## CHAP. V.

### OF THE REDUCTION OF FRACTIONS TO THEIR LOWEST TERMS.

1. SINCE the value of fractions is most easily ascertained when they are at their lowest terms, it is of advantage when feasible to divide fractions by a common measure. The greater that common divisor may be so much less will be the quotients or terms of the fraction equal to the given one. It is necessary, therefore, when two numbers are given, to have a method of finding their greatest common measure, that is to say, the greatest divisor which will divide the given divisor without a remainder. Such is the following:

2. Divide the greater of the given numbers by the less, and that divisor by the remainder of the division; and, if still there be a remainder, you should by it divide the former divisor—that is, the last remainder, and so on until you come to a divisor which exhausts or measures its dividend, that is, the greatest common measure of the two.

For instance, let 9 and 15 be the given numbers. I divide 15 by 9, and 6 remains; I divide 9 by 6 and 3 remains; I divide 6 by 3 and nothing remains. Therefore 3 is the greatest common measure of the two numbers 9 and 15, which I show thus.

(*a*) 3 measures 6, but (*b*) 6 measures 9, if 3 be taken away. Therefore 3 measures 9, if 3 be taken away; but 3 measures itself, therefore it measures the whole 9; but (*c*) 9 measures 15, 6 being taken away; therefore 3 measures 15, 6 being taken away; but it measures 6, therefore it measures the whole number 15. Hence it is clear, that 3 is the common measure of the given numbers, 9 and 15. It remains for me to show, that it is the greatest common measure. If not, let there be some other common measure, say 5. Now since (*e*) 5 measures 9, (*d*) but 9 measures 15, 6 being taken away; it is plain that 5 measures 15, 6 being taken away; but it measures the whole 15 (by hypothesis), therefore it measures 6; but 6 measures 9, 3 being taken from it; therefore 5 measures 9, 3 being taken from it. Therefore since 5 measures both the whole of 9, and 9, 3 being taken from it: it will also measure 3 itself, that is, (*f*) the lesser number, which is absurd.

The greatest common measure being found, it is plain that the fraction $\frac{9}{15}$ can be lowered to this fraction $\frac{3}{5}$, which I thus show to be equal to the former. Every fraction denotes the quotient of the numerator divided by the denominator; but in division the quotient expresses the ratio of the dividend to the divisor; whilst the ratio therefore remains the same, the quotient or frac-

(*a*) By construction. (*b*) By construction. (*c*) By construction. (*d*) By construction. (*e*) By hyp. (*f*) By hyp.

tion will be the same. Moreover, it is very clear, that the ratio is not changed, its terms being equally divided; for instance, if any thing be double of another, or triple, the half of that will be double or triple of the half of the other.

Those who can divide and multiply fractions by integers will find no difficulty in reducing fractions of fractions to integers. For instance, this fraction of a fraction $\frac{3}{4}$ of $\frac{2}{5}$, what else is it, than three times the fourth part of the fraction $\frac{2}{5}$, or $\frac{2}{20}$ mutiplied into the integer 3? In like manner, the numerators and denominators being mutually multiplied into the fraction of a fraction of a fraction, is reduced to an integer. Since these things are so clear and manifest, it is amazing by what circuitous processes, what a tedious apparatus of theorems, quotations, and species they are demonstrated, or rather obscured.

# ARITHMETIC.

## PART III.

### CHAP. I.

#### OF THE RULE OF PROPORTION.

THE rule of Proportion is that by which, three numbers being given, a fourth, proportional to them, is found. Its use is frequent and very great, and hence it is called the golden rule. It is also called the Rule of Three, on account of the three given terms. We directly find the fourth proportional by multiplying the second term by the third, and dividing the product by the first; for instance, if as 2 is to 6, so should 4 be to the number required; multiply 4 into 6, and divide the product 24 by 2, the quotient 12 will be the fourth proportional required, which I demonstrate as follows.

In four proportionals the product of the extremes is equal to the product of the intermediate terms. For, since the numbers are proportional, that is, have the same ratio between themselves, but ratio is estimated by division, if the second term be divided by the first, and the fourth by the third, the quotient will be the same, which, according to the nature of division, multiplied into the first term will produce the second, and into the third will produce the fourth. If therefore we multiply the first term into the fourth, or which is the same thing, into the third and common quotient, and the third term into the second, or which is the same thing, into the first and common quotient, it is clear that the products will be equal, as the factors are in each case the same. But from the nature of multiplication and division, it is clear that the product being divided by one of the factors, the other is the quotient; therefore if I divide the product of the two intermediate terms (6 and 4) by the first (2), the quotient (12) will be the fourth proportional sought.

Question 1. A traveller in 3 hours goes 15 miles: how many will he go in 9 hours? Answer 45. For it is clear from the question, that as 3 is to 15, so is 9 to the number required; that is, 3 : 15 : : 9 : therefore 135, the product of 9 into 15, divided by 3, will give the number required, that is 45.

Question 2. If 2 workmen in 4 days earn 2s., how much will 5 earn in 7 days? that is, as 2 multiplied by 4 are to 2, so are 5 multiplied by 7 to the number sought; or as 8 : 2 : : 35 to the number sought; and thus the hire sought is found to be 8s. 9d.

Question 3. Three merchants forming a partnership, gain £100. The first spent £5, the second £8, the third £10. It is sought how much each gained. The sum of the expenditure is £23. Say therefore, as 23 is to 5, so is £100 to the sum sought. The resulting number will indicate how much is due to the first from the common gain, for it is fair that, as the expense of each is to the sum of the expenses, so should be his gain to the sum of the gains. Further, in the same way by saying 23 : 8 : : 100 &c., and 23 : 10 : : 100, &c., the gains of the others will appear.

The inverse rule of proportion is easily resolved into simple. For instance, 2 men expend £5 in 6 days: in how many days will 8 men expend £30? Say first 2 : 5 : : 8 : &c. and the answer will be 20; say therefore, then as 20 : 6 : : 30: &c. and you will find the number required. It is superfluous to explain, why the sought term immediately is found by means of this rather intricate rule.

4th Question. Four pipes fill a cistern in 12 hours: in how many hours will it be filled by 8 of the same size? We should say, as 8 : 4 : : 12 : &c.; then 4, multiplied into 12, that is, 48 divided by 8, give the answer, that is 6. Nor in this case, when the proportion is inverted, is there any new difficulty; for the terms being properly arranged, we shall have two equal rectangles of one, of which both sides are known, but the other is produced from the known term, multiplied into the unknown; and by dividing that first product by the known side or factor, the unknown term will come out. But it will appear from the question itself, in what order the terms are to be arranged.

## CHAP. II.

### ON ALLIGATION.

THE rule called Simple Alligation is that by which, two things being given, of different price or weight, &c., there is found some third sort, so compounded of the given, that its price, weight, &c., be equal to a given price, weight, &c., intermediate between the given. For instance, a cubic inch of gold weighs 18 ounces, a cubic inch of silver weighs 12 ounces. It is required to have a cubic inch of metal compounded of both, and weighing 16 ounces: in which problem, the intermediate weight 16, exceeds the weight of silver by 2, and is exceeded by the weight of gold, by 2. Now if we take $\frac{2}{6}$ of a cube inch of silver and $\frac{4}{6}$ of a cube of gold, it is clear that these, if combined, will make up a cube inch, as they are equal to unity. But it is also plain, that the weight of this mixed metal is equal to the intermediate 16; for we took 2 parts of silver, which is lighter by 4, therefore the defect is 2 in 4; but of gold, which is heavier by 2, we took 4 parts, so the excess is 4 in 2; that is, equal to the defect, so that they counterbalance each other.

Hence results a rule for the alligation of two things. The quantity of the greater, which is to be taken, is indicated by a fraction, the denominator of which is the sum of the differences, and the numerator, the difference between the middle and less; and in the next place, that which has the same denominator, and for numerator the difference between the greater and middle, shows the less quantity which is to be taken.

Question. There are two kinds of silver: the ounce of finer is worth 7 that of inferior quality is worth 4; we want to find 3 ounces of silver which are each worth 5. Answer. It is clear, that if I take $\frac{2}{3}$ of the inferior, and $\frac{1}{3}$ of the finer, there will be one ounce of the mixed compound, and three times this will be the quantity required.

If the things to be alligated are more than two, the rule is called Compound Alligation. For instance, there are five kinds of wine, the strength of massic is 1, of chian 3, of falernian 5, of cæcuban 7, of corcyræan 9: I require a mixture, the strength of which is 4. The strength of a mixture of equal parts of massic and chian, will be 2, being half of the sum of that of the given quantities 1 and 3, as is plain; so the strength of a mixture of equal parts of falernian, cæcuban, and corcyræan will be 7; that is $\frac{1}{3}$ of the number 21, or of the sum of the strengths of the components of the mixture. I alligate 2 and 7 with the given intermediate strength 4, and the defect is 2, the excess 3, the sum of the differences 5; consequently there should be taken $\frac{3}{5}$ of the first mixture, $\frac{2}{5}$ of the latter, then $\frac{3}{5}$ being divided by 2, the quotient shows how much of each, chian and massic, should be taken. In the same way $\frac{2}{5}$ divided by 3, will indicate how much of falernian, &c., should be in the required mixture. So $\frac{3}{10}$ massic, $\frac{3}{10}$ chian, $\frac{2}{15}$ falernian, $\frac{2}{15}$ cæcuban, $\frac{2}{15}$ corcyræan, will give the answer.

Hence we perceive how compound alligation may be reduced to simple. The prices, magnitudes, weights, or whatever else should be alligated, ought to be collected into two sums, which are to be divided each by the number of terms which constitute it; the quotients should be alligated with the intermediate term; the resulting fractions, divided each by the number of things entering into the sum to which they refer, will express the quantity of each to be taken. The demonstration is plain, from what has been said.

N.B. In alligation of several things, each question admits of innumerable solutions, and this for two reasons; first, the deficient terms can be combined with the exceeding ones in several ways, whence will result various quotients to be alligated with the given intermediate term. Care should be taken, however, lest the quotients be together greater, or together less, than the mean; for if this happen, it is plain that the question is impossible; secondly, it is allowable frequently to repeat the same term, whence its share or portion will be increased, but those of the others diminished.

I am glad here, for the gratification of the studious, to give a solution of that famous problem given to Archimedes by Hiero.

Question: A crown is made of an alloy of gold and silver: it is asked how much gold, how much silver, is in it, and the king does not allow the crown to be broken up. Ans. Two masses should be taken, one of gold, another of silver, each of equal weight with the crown, which being done, it is manifest that the problem could be proposed in another form, as follows: a pound of gold and a pound of silver being given to find a pound of an alloy made up of both, which shall be of the given intermediate mass. Now, as the solid contents of the crown cannot be ascertained geometrically, there is need of contrivance. Each of the masses should be separately immersed in a vessel full of water, and the quantity of water which flows out on each immersion should be measured, it being obvious that it must be equal in bulk to the immersed mass; suppose the gold being immersed, let the bulk of displaced water be 5, of the silver 9, of the crown 6. The question, therefore, comes to this; there being given a pound of gold of the magnitude of 5, and a pound of silver of the magnitude of 9, it is required how much of each we should take to have a pound of an alloy of the magnitude of 6. Then if 9 and 5 be alligated with the intermediate magnitude 6, the quantity of gold will be ascertained, that is, $\frac{3}{4}$ of the quantity of gold, and $\frac{1}{4}$ of the quantity of silver, combined in the crown.

Hence it appears how little difficulty there is in the problem on the solution of which Archimedes of old exclaimed Ἕυρηκα.

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## CHAP. III.

### OF ARITHMETICAL AND GEOMETRICAL PROGRESSION.

ARITHMETICAL progression is a name given to a series of numbers increasing or diminishing by a common difference. For instance, in this series 1, 4, 7, 10, 13, 16, 19, 22, 25, 3 is the common excess by which the second term exceeds the first, the third the second, the fourth the third, and so on; and in this other series of decreasing numbers 15, 13, 11, 9, 7, 5, 3, 1, 2 is the common quantity by which each number falls short of the preceding.

Now, from considering this series and the definition which we have laid down, it is manifest that each term contains the lesser extreme, and the common difference multiplied by the number of places by which it is removed from that lesser extreme. For instance, in the first series the number 13 consists of the lesser extreme 1, and the common difference 3 multiplied into 4, the number of places by which it is removed from the lesser extreme. Hence the lesser extreme and the common difference being given, any term, for example the eleventh from the lesser exclusive, can be easily found, by multiplying the difference 3 into 11, and adding the product 33 to the lesser extreme 1. If the greater extreme, the common difference, and the number of places intervening between the term sought and the greater extreme be given, the term sought may be found by multiplying the difference into the number of places, and taking the product from the greater extreme. It is clear also how the first term is assigned, if any term, its index, and common difference be given; and how the common difference may be obtained if any term, its index, and the lesser extreme be given; and also how the index of any term may be obtained if the term, the difference, and lesser extreme, be given. It is also clear that the half of the sum of two terms is equal to the arithmetical mean proportional. For instance, 7 and 13 make 20, whose half, 10, is a mean between the given terms. Any one can easily deduce these and many other problems and theorems, and their solutions, from the nature of arithmetical progression, especially if he use skilful symbolical computation. I therefore leave them to beginners for points of practice.

GEOMETRICAL progression is the name given to a series of numbers increasing or decreasing by the same continued ratio. For instance 3, 6, 12, 24, 48, 96 are in a geometrical progression, the common ratio of which is twice, as each term is twice the preceding one. In like manner the numbers of this decreasing series 81, 27, 9, 3, 1, proceed in a subtriple ratio, that is, each term is in a subtriple ratio to the preceding, or $\frac{1}{3}$; where it should be observed that each term is composed of the power of the common ratio, bearing the same name with it, multiplied into the first term. For instance, 48, the fourth term, exclusively, is produced from 16, the fourth power of the number 2 (that is, that which produced from 2 multiplied three times into itself, since the root itself is called the first power,) multiplied by the first term 3. Wherefore what has been laid down concerning arithmetical progression holds good here, if, instead of addition and subtraction, we use multiplication and division; and instead of multiplication and division, involution and evolution, or the extraction of roots.* For instance, as in arithmetical progression the sum of the extremes divided into two equal parts gives the arithmetical mean, so in geometrical progression the mean proportional is the root of the product of the extremes. Therefore, as regards theorems and problems, we shall not longer dwell upon deducing them, since they easily result from the mere consideration of the series.

But there is one theorem of geometrical progression from which the knowledge of logarithms was originally derived, and on which they still rest; and which therefore it is fit to explain here. In a geometrical progression, the commencement of which is unity, the rectangle of any two terms is equal to the term of the same progression, which has for index the sum of the index of the factors. For instance, if in the following series, $\left\{\begin{matrix} 1, & 2, & 4, & 8, & 16, & 32, & 64, \\ 0, & 1, & 2, & 3, & 4, & 5, & 6, \end{matrix}\right\}$ the second term 2 be multiplied into the fourth 8, the product, 16, is the fifth term, the index of which, 4, is the sum of the indexes of the second and fourth. The reason is manifest; for each power, multiplied into any other power of the same root, produces a third, in which there are as many dimensions as there were in both of the generating powers. But in a geometrical progression, of which the first term is unity, it is clear that all the other subsequent ones are powers produced from the common

* N.B. A careful reader can investigate how the roots of any powers may be extracted by means of the method which we followed when treating of the cube, the square, and their roots.

ratio, and that each have as many dimensions as their places are distant from unity. Therefore, if to an infinite geometrical progression there were annexed also an infinite series of indices, to obtain the rectangle of two terms it would not be necessary to multiply one by the other; but, merely adding the indices to find an index equal to the sum, and this would show the sought rectangle annexed to it. In like manner, if one term is to be divided by another, the difference of the indices, if the square or cube root is to be extracted, $\frac{1}{2}$ or $\frac{1}{3}$ as an index, would show the required quotient or root.

Hence it is plain that the more difficult operations in arithmetic could be singularly abridged, if tables were formed in which the numbers placed in natural order should have each a corresponding index annexed; for then multiplication can be effected by addition alone; division by subtraction; the extraction of roots by halving or trisecting the indices: but to accommodate those indices or logarithms to numbers, "this is the task, this the labour," in effecting which numbers of mathematicians have toiled.

The first who formed tables generally proceeded in this way. To the numbers 1, 10, 100, 1000, &c., in decuple progression, they assigned the logarithms 0,0000000, 1,0000000, 2,0000000, 3,0000000, &c. Then to find the logarithm of any number, for instance, 4, between 1 and 10, seven cyphers being added to each, they sought a mean proportional between 1,0000000 and 10,0000000; and if it were less than 4, the mean proportional was to be sought between it and 10,0000000, but if greater, between it and 1,0000000; and then they finally sought a mean proportional between this (if it were less than 4) and the next greatest, but if greater, and the next less, and so on until they came to a number differing by a very small part, for instance $\frac{1}{100000}$, from the proposed 4. Then the logarithm of this was had by finding an arithmetic mean between the logarithms of the numbers 1 and 10, and another between it and the logarithm of ten, &c. Now if the logarithm of the number 4 be halved, the logarithm of 2 is found; the same when doubled gives the logarithm of the number 16; and if to the logarithm of 4 be added the logarithm of 2, the sum will be the logarithm of 8.

In the same way, from one logarithm of the number 4, others innumerable can be obtained. In this manner, when logarithms were adapted to other numbers between 10 and unity, they supplied very many others to their sum and difference. But of this enough; for we have not undertaken to give all things which bear on logarithms. It was merely proposed to a certain extent to explain their nature, use, and invention.

# MATHEMATICAL MISCELLANIES:

SOME THOUGHTS

CONCERNING

SURD ROOTS, THE ATMOSPHERIC TIDE, AN EQUILATERAL CONE AND CYLINDER, CIRCUMSCRIBED ABOUT THE SAME SPHERE, ON THE ALGEBRAIC GAME,

AND SOME PERSUASIVES TO

THE STUDY OF MATHEMATICS, ESPECIALLY ALGEBRA.

BY * * * * BACHELOR OF ARTS,

TRINITY COLLEGE, DUBLIN.

TO THAT EXCELLENT YOUTH,

# D. SAMUEL MOLYNEUX,

FELLOW COMMONER IN DUBLIN COLLEGE, SON OF THE EMINENT WILLIAM MOLYNEUX, CUT OFF A FEW YEARS AGO, BY A FATE LAMENTABLE BOTH FOR HIS COUNTRY AND THE INTERESTS OF LITERATURE.

Excellent Youth,

Such was the esteem in which your father when living was held by the learned, that I consider I would do them a grateful service, if I should show that he has left a son, who inherits his penetration and sagacity. It must indeed be allowed, that your uncle,* a man of a remarkably enlarged and well-informed mind, had previously done something of the kind. For that eminent man had perceived your disposition as you approached maturity, that it was probable you would follow in the footprints of your father. The authority of such a man influenced me so far, that I from that thenceforth conceived great hopes of you. But now when becoming acquainted with the nature of your studies, I perceive you devoting yourself to sound philosophy and mathematics, when I perceive that the thorns which seem to beset mathematics, and usually deter others from its study, on the contrary, spur you on to a more speedy progress; when I also see, that high intellectual powers accompany that industry and desire of knowledge, I cannot restrain my joy from manifesting itself to the learned world, and expressing my undoubting anticipation that, if God grant you life and health, you will be one of the chief ornaments of the rising age. Wherefore, presenting to you the following pieces, whatever may be their merit, I wished to seize the opportunity of communicating publicly with, as well that I might gratify my affection towards you, as well as, that expectation being raised respecting you, you might by some tie, and that no ungrateful one, be attached to the study of such excellent objects.

* See a letter of Thomas Molyneux, M.D., to the Bishop of Clogher, in Philosoph. Transac. No. 282.

# OF SURD ROOTS.

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It formerly occurred to me to think that algebraical operations would be rendered more easy, if, discarding the radical signs, some other method could be contrived of computing the roots of imperfect powers which would be less at variance with the forms used in other processes. For, as in arithmetic, fractions are rendered much more manageable by being reduced from vulgar to decimal, for then, the place of each figure serving as a denominator, they are abridged in one part and expressed as integers, and form the same sort of series as they do, and are regulated by the same rules: so if from symbolical computations we removed that radical sign ($\surd$), which, as the denominator marks the difference between integers and fractions, points out a difference in treating radicals and surds, unquestionably the mode of treating them would be simplified.

Wherefore should we not, therefore, designate surd roots like rational, merely by letters, and substitute $c$ or $d$ for $\sqrt{b}$? for if they were expressed in this way there will be no distinction between them and the roots of perfect powers, and addition, subtraction, multiplication, &c., will be managed in the same manner in each case. But there is a ready objection, that quantities multiplied in this way confuse calculations more than radical signs do. To this I answer, a remedy can be applied if we use the letters of the Greek alphabet for expressing roots, by writing $\beta$ for $\sqrt{b}$, $\gamma$ for $\sqrt{d}$, &c.; in which way not so much the letters themselves as the characters will be varied, and each substituted figure will so far correspond to the primitive that there will be no room for scruple. The root of a quantity produced from the multiplication or division of others, will be marked by their roots simply multiplied and divided, for instance, $\sqrt{bc} = \beta\kappa$, and $\sqrt{\frac{bdm}{e}} = \frac{\beta\delta\mu}{\epsilon}$

But if a multinomial quantity, or one consisting of many members, in which there is no unknown quantity connected by the signs + or —, be proposed, their aggregate might be expressed (as is often the case) by some one letter. For instance, let $a+b-c=g$, the root of which is $\gamma$. You will perhaps ask, what is to be done where unknown and known quantities are connected; for instance, if the imperfect power be $f + x$: for if we use $\phi$ and $\xi$ as the roots of the parts, the root of the whole cannot be determined from them. Why, then, could we not render the given imperfect power equal to some perfect one; as for instance,

$f + x = ff + 2f\,\overline{\xi + \xi\xi}$, or $ff\overline{f + 3f}\,\xi + 3f\,\xi\xi + \xi\xi\xi$, &c.? For then $f + x = \sqrt[2]{f + x}$, or $\sqrt[3]{f + x}$, &c.

But it has been omitted how we are to ascertain of what sort is the root; whether quadratic, cube, or biquadratic; and whether, Greek characters being left to quadratics, others should be used for the rest. Or rather, the characters remaining the same, we should by means of one point placed above, denote a square root, by two a cube root, by three a biquadratic root, &c.; in the same way as first, second, third, &c., fluxions are expressed. Or, finally, should consider it sufficient that the denominator of the root might appear by retrogression; since in the course of the operation it is of no import of what kind the root may be, since all expressed without a radical sign are subject to the same laws, and treated in the same manner.

These things are, indeed, crude and imperfect; and I am aware of how little value is that which I am proposing. But you, illustrious youth, who have both leisure and abilities, can perhaps extract some good from this refuse. However I am not certain whether the thing which we have been discussing may not be of some use to beginners, for I know others will little regard them; and whether by their aid the thread of analytic investigation may not be disentangled when the radical sign being laid aside, the heterogeneous operations which accompany it may also disappear. However this may be, I am convinced that these being partially set forth, I could more briefly and clearly explain the common theory of surds than I am aware that it has been done by any one. I now proceed to do it.

Surd roots are said to be commensurable when their mutual ratio can be expressed by rational numbers, but if this cannot be done they are called incommensurable. If there be given two surd roots, and it be required to ascertain whether they be commensurable or not, let there be found an exponent of the ratio existing between the powers to which the radical sign is prefixed; if this be a perfect power, having the same index as the proposed roots, they will be commensurable; but otherwise, then they should be regarded as incommensurable. For instance, let the proposed roots be $\sqrt[2]{24}$ and $\sqrt[2]{54}$, the fraction $\frac{4}{9}$ squared, expresses the ratio of one power, 24, to the other, 54; and consequently the roots are commensurable: that is to say, $\sqrt[2]{24} : \sqrt[2]{54} :: 2 : 3$. Let $\sqrt[3]{320}$ and $\sqrt[3]{135}$ be the given quantities, the ratio of the number 320 to 135 is expressed by $\frac{64}{27}$, a perfect cube; the root of which, $\frac{4}{3}$, indicates the ratio of the one root, $\sqrt[3]{320}$, to the other, $\sqrt[3]{125}$. The demonstration is manifest, since all know that square roots are in subduplicate ratio, cube roots in subtriplicate, biquadratic in subquadruplicate, and so on of the respective powers.

If the roots, the ratio of which is required, be heterogeneous, they should be reduced to one kind by involving the numbers affixed to the radical sign, each according to the index of the other root, which being thus involved, the radical sign is to be prefixed, with an index produced by multiplying together the indexes originally given. For instance, let the heterogeneous surd roots be $\sqrt[2]{5}$ and $\sqrt[3]{11}$; if 5 be cubed and 11 be squared, they will become 125 and 121. The radical sign, prefixed with the index 6, produces the homogeneous roots $\sqrt[6]{125}$, $\sqrt[6]{121}$. That the reason of this operation may be perceived, let $\sqrt[2]{5}$ be denoted by some simple sign, suppose $b$, and $\sqrt[3]{11}$ by $c$, and $\sqrt[2]{bb}$ will be $= \sqrt[2]{5}$, and $\sqrt[3]{ccc} = \sqrt[3]{11}$, and $\sqrt[6]{bbbbbb} = \sqrt[6]{125}$, and $\sqrt[6]{cccccc} = \sqrt[6]{121}$; where it is plain that $\sqrt[6]{bbbbbb} = \sqrt[2]{bb}$, $\sqrt[6]{cccccc} = \sqrt[3]{cc}$.

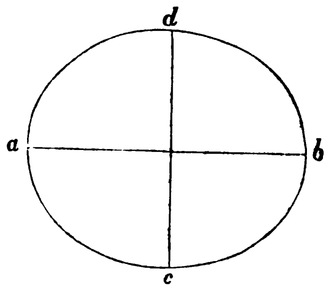
As to the addition of surd roots, if they are commensurable it is done by prefixing the sum of the terms to the radical sign of the ratio, under which the common divisor is to be placed, by means of which the terms of the common ratio were denoted. For instance, $\sqrt[2]{24} + \sqrt[2]{54} = 5\sqrt[2]{6}$. For, from what has been already observed, and from what follows concerning multiplication, $\sqrt[2]{24} = 2\sqrt[2]{6}$, and $\sqrt[2]{54} = 3\sqrt[2]{6}$. In the same way is subtraction managed, only that the difference of the terms is prefixed to the radical sign of the exponent. If incommensurable surd roots are to be added or subtracted, they should be connected by the signs + or —. For instance, $\sqrt{6} + \sqrt{3}$ and $\sqrt{6} - \sqrt{3}$, are the sum and difference of the roots of the numbers 6 and 3; in which way, also, rational numbers are added to or subtracted from surds. If the surd root is to be multiplied by another homogeneous, the radical sign and the common index should be prefixed to the rectangle of the powers. For instance, the $\sqrt[2]{3} \times \sqrt[2]{7} = \sqrt[2]{21}$, and $\sqrt[3]{g} \times \sqrt[3]{x} = \sqrt[3]{gx}$. For demonstrating which operation, let the roots of the numbers 3 and 7 be denoted by $b$ and $d$, so that $bb = 3$ and $dd = 7$, and it is manifest that $\sqrt[2]{bbdd} = bd$; that is, the square root of the product is equal to the products of the square roots. The same thing can be demonstrated in the same way concerning any other roots, cubic, biquadratic, &c. Heterogeneous roots, before they are multiplied, should be reduced to homogeneous. If a rational number is to be multiplied into a surd, it should be raised to a power of the same denomination with the given imperfect one, to which is prefixed the radical sign and the index of the same power, and then proceed as before. For instance, $5 \times \sqrt[3]{4} = \sqrt[3]{125} \times \sqrt[3]{4} = \sqrt[3]{500}$; or more compendiously thus: $5\sqrt[3]{4}$, and generally $b \times \sqrt[a]{c} = \sqrt[a]{b^a c}$, or $b\sqrt[a]{c}$.

As to division, as often as the dividend and divisor are both surd roots, having removed what is heterogeneous, if there be any, the radical sign prefixed to the quotient of the powers, with the proper index, will exhibit the required quotient. For instance $\sqrt[2]{7} \div \sqrt[2]{3} = \sqrt[2]{\frac{7}{3}} = \sqrt[2]{2\frac{1}{3}}$. But if of two numbers only one be under the radical sign, the other, involved according to the index of the given root, should be placed under the radical sign, and proceed as before. For instance, $\sqrt[3]{96} \div 4 = \sqrt[3]{96} \div \sqrt[3]{64} = \sqrt[3]{\frac{96}{64}} = \sqrt[3]{\frac{3}{2}}$; or without preparation, $\frac{\sqrt[3]{96}}{4}$; and generally $\sqrt[a]{c} \div b = \sqrt[a]{\frac{c}{ba}}$ or $\frac{\sqrt[a]{c}}{ba}$. These are, as well as the former, easily demonstrated.

## ON THE ATMOSPHERIC TIDE.

Some time since I met with a book bearing the title, "On the Influence of the Sun and Moon on the Human Body," by an eminent doctor of medicine and F.R.S. I am well aware how celebrated he is and how insignificant I am. But that I may freely declare my opinion, I thoroughly receive the opinion concerning the atmospheric tide as he there explains it, and how it is based on the principle of the celebrated Newton. I am not, however, convinced that the ingenious author has rightly ascertained the causes of some phenomena connected with it. How far my doubts may be well founded, you, with whose acuteness I am well acquainted, will best judge.

That eminent man considers that there is a swelling out of the spheroidal figure of the earth about the equinoctial line. He attributes to the same cause the difference between the swelling of the air caused in the oblique sphere by the meridional, and (if I may use the expression) antimeridional moon. But I do not think that the explanation of either of those phenomena should be sought in the oblate spheroid. On this account, because though the opinion that the aëreo-terrestrial mass is of that figure is supported both by mathematical and physical grounds, and also agrees well with some phenomena, still it is not so fully received by all, but that some, and those of note too, hold the opposite opinion. And indeed I remember that Mr. Chardellou, who is profoundly skilled in astronomy, informed me that he had ascertained that the axis of the earth is longer than the diameter of the equator, and consequently, that the earth is a spheroid, but such as Burnet describes it, rising at the poles and lower at the equator. But as for me, I would rather call in question the observations of that eminent man, than reject the arguments for the earth being oblate. Still since that opinion does not equally please all, I should be reluctant to adopt it as a principle for explaining any phenomenon, unless the thing could not otherwise be explained. But in the next place, so far from the explanation of the above-mentioned effects requiring necessarily a spheroidal figure of the earth, that it gains not a particle of light from it, and I will try to show this, by adding what that eminent man writes on the subject: "The air rises above its usual level about the two equinoxes, because when the equinoctial line corresponds with that circle of the terrestrial globe which has the greatest diameter, each of the heavenly bodies, while in it, is nearer to the earth." On the Influence of the Sun and Moon, p. 9. However, it may be well doubted, whether that nearer position of the luminaries be adequate to raise the air above the usual level. For so slight is the difference between the transverse and conjugate axis of the ellipse, by the revolution of which the terrestrial spheroid is generated, that it approaches very near to a sphere. But that we may consider the question more accurately, let *a c b d* denote a section of the aereo-terrestrial mass through the poles, *d c* being the axis, *a b* the diameter of the equator. Now by calculation I have ascertained, that the attractive power of the moon is not $\frac{1}{4000}$ part stronger at *a* or *b* than it would be at *c* or *d* if it directly were above either pole,



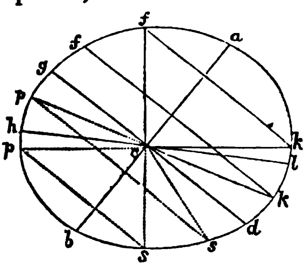

and, therefore, that so small a difference would be altogether unequal to producing any sensible effect. It should also be considered, that the moon is never distant from the equator a third part of the arc *b d*, and that consequently the aforesaid difference must be still very much restricted. But what we have said of the moon must be still further the case as to the sun, since it is many times further distant. It is true indeed that Dr. Mead has adduced, besides other causes of the tide being higher near the equinoxes, to wit, "the greater agitation of a fluid spheroid revolving in a greater circle, besides the centrifugal force having a much greater influence there." As to the first, although at first it appears of some import, I must confess that I do not altogether perceive how any thing bearing on the distinct explanation of the subject can be concluded from that. As to the second, it is plain that the centrifugal force is far the greatest near the equator, and on that account that the aereo-terrestrial mass has the figure of an oblate spheroid; but I do not see what consequence results from this. But although we should allow that the air, for the causes mentioned by this eminent man, should about the equinoxes swell out near the equator, it does not, however, appear thence, how with us, who live so far from the equator, it then should be higher than usual, but rather the contrary seems to follow. Mead writes thus in the following page: "To conclude, in the same parallels where the declination of the moon is towards that pole of the heavens which rises highest, the attraction is the strongest, when it comes to the meridian of the place, but least, when it comes to the meridian of the opposite place; the contrary of which happens in the opposite parallels. The cause is in the spheroid figure of the earth and atmosphere." But I do not think that the cause is in the figure of the earth and ambient atmosphere, because if we assume the earth to be perfectly spherical or even oblong, the same thing will certainly happen, as will be shown below.

It remains for me to attempt the explanation of these things, especially on this account, because a reason drawn from the spheroidal figure of the earth was regarded by me with suspicion; for without taking it at all into account, the affair could be most clearly and easily explained.

Newton, in his Physico-mathematical work, book iii., prop. 24, where he explains the phenomena of the tides of the sea, has this passage: "The influence of each luminary also depends on its declination, or distance from the equator. For if the luminary were placed at the pole it would constantly attract every particle of water without increase or diminution of its action, and so would cause no reciprocation of motion. Therefore the luminaries, in proceeding towards the pole from the equator, will gradually lose their effects, and on that account will cause less tides in the solstitial than in the equinoctial syzygies." But no other cause need be sought for any phenomenon of the atmospheric tide, than is sufficient for producing a similar effect in the tide of the sea. But that I may explain more fully that which has been rather briefly, and therefore obscurely by the most eminent man in the world, in the former figure let *a d b c* be the meridian, and *a b* the axis of the aereo-terrestrial mass, and let the sun and moon be conceived to be placed at the poles. It is clear that each part of the aerial mass, *d* for instance, during the diurnal revolution retains an equal distance from the luminary, and so is equally attracted towards their bodies; so that the air is not at one time elevated, at another depressed, but through the whole day remains at the same altitude. But secondly let *a c b d* represent the equator, or some parallel, and let the luminaries be in the equinoctial plane; at that time it is plain that the equator itself, as well as each parallel, assumes an elliptical figure. It is manifest also that the air which now is at *a*, the summit of the transverse axis, and is the highest six hours afterwards, will be at *c*, the extreme of the conjugate axis and lowest, and that the greatest reciprocation of motion results.

To finish the whole work at once, let us suppose the swellings of the tidal spheroid to have a threefold position, either in the poles, or in the equator, or in the intermediate parts. In the first case, the plane of diurnal rotation would be perpendicular to the axis of the spheroid, and therefore a circle; whence there would be no tide; in the second, it would be parallel to the same, and consequently an ellipse, between the axes of which would be the greatest difference, consequently the tides would be greatest; in the third, in proportion as it approached nearer to the perpendicular position, it would be more nearly a circle, and consequently the tides would be less.

It remains that I should demonstrate that the difference which exists in an oblique sphere between any tide and the following one, when the moon is away from the equator, will result indifferently, the earth being assumed either oblate, or exactly spherical, or oblong. Let *a b* be the axis of the world, *g d* the equator, *k* any place, *f k* the parallel of the place, *h l* the axis of the tidal spheroid, swelling on both sides principally by the influence of the moon. Let the moon's place be near *l*. It is to be demonstrated that *c k*, the height of the air when the moon is near the meridian, is greater than *c f*, the height of the air when the moon has passed the meridian of the opposite place. Let *p s* be drawn, a parallel corresponding to the former on the opposite side, and let *c k c f* be produced to *p* and *s*. By construction the arc *p h* is equal to the arc *k l*, therefore the arc *f h* is greater than the arc *k l*; therefore on account of the ellipse the right line *f s* is less than the right line *k p*, and *f c* less than *k c*. Q. E. D.

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## OF THE EQUILATERAL CONE AND CYLINDER DESCRIBED ROUND THE SAME SPHERE.

### LEMMA.

The side of an equilateral triangle is to the diameter of the inscribed circle as $\sqrt{3}$ to 1, and the perpendicular, let fall from any angle to the opposite side, is to the same as 3 to 2.

These things are plain to any one at all acquainted with algebra and geometry.

### PROBLEM.

To find the ratio between the cylinder and equilateral cone circumscribed about the same sphere.

Let the diameter and periphery of the base of the cylinder be each unity. Then by lemma the diameter and the periphery of the base of the cone will be each $\sqrt{3}$. Then $1 \times \frac{1}{4} = \frac{1}{4} =$ base of cylinder; and $\frac{1}{2} =$ the sum of the bases. And $\sqrt{3} \times \frac{1}{4}\sqrt{3} = \frac{3}{4} =$ base of the cone, and surface of the cylinder, or four times the base is equal to 1. And the simple surface of the cone is equal to $\frac{3}{2} = \frac{\sqrt{6}}{4} \times \sqrt{6}$; for $\sqrt{\frac{3}{2}}$ (that is, a mean proportional between $\sqrt{3}$, the side of the cone, and radius of the base or $\sqrt{\frac{3}{4}}$) is the radius of a circle equal to the surface of the cone; and by the preceding $1 + \frac{1}{2} = \frac{3}{2} =$ surface of whole cylinder, and $\frac{3}{2} + \frac{3}{4} = \frac{9}{4} =$ surface of whole cone. Consequently by lemma and hypothesis the axis of the cylinder is 1, and of the cone $\frac{3}{2}$. But the solid contents of the cylinder $= \frac{1}{4} \times 1 = \frac{1}{4}$, and the solid contents of the cone $= \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$. Hence, the homogeneous quantities being compared together, there will result the following.

### THEOREM.

Between an equilateral cone and cylinder circumscribed about the same sphere, there is the same sesquialterate ratio as to the whole surfaces, the solids, altitudes, and bases. Two years ago I discovered that theorem to my no small surprise. I did not wonder at my own talents or peculiar sagacity, as the thing is so easy, but merely that Tacquet, a celebrated professor of mathematics, prided himself so much on a discovery, to which a beginner is competent. His discovery, which is but a part of that stated above, is, "that an equilateral cone is sesquialterate in solid contents and entire surface, of a cylinder circumscribed about the same sphere," and that consequently, "there is a continued ratio" between an equilateral cone, cylinder, and sphere.

This is that proposition, to which reference is made by the figure, which is with an inscription placed on the title page of that author's work on the select theorems of Archimedes. The reader may still further consult what the Jesuit states in his preface, in the scholium to Prop. 32, and at the end of the 44th proposition of the same work, where he puts forward his theorem as a wonderful invention, and rivalling those of Archimedes. And not only Tacquet, but the celebrated Wallis also, brings it out in the additions and emendations to the 81st chapter of his Algebra, as having been demonstrated by Caswell by means of the arithmetic of infinites; which is also done, as to one part, by

Dechales, in the 20th proposition of his book about indivisibles. However, the method of indivisibles and the arithmetic of infinites founded on it, are by some scarcely allowed to be geometrical.

But the whole theorem has been demonstrated before by no one that I know. However, if it be true, as Tacquet thinks, that amongst the other various and celebrated discoveries of Archimedes, he was most pleased with that in which he shows that the cylinder is sesquialterate of the inscribed sphere in solid contents and surface, because there is one connected proportion of the bodies and of the surfaces containing them. If this was the reason that he wished the cylinder circumscribed about the sphere to be sculptured on his tomb, what would the old Sicilian have done had he discovered that the one connected proportion held in a fivefold respect as to the two bodies. But we have just noticed how easily that follows from his inventions.

In much the same way as we have done that, it will not be difficult to discover and demonstrate all the theorems which Tacquet annexes to the Archimedean, and a hundred more of the same sort, if necessary.

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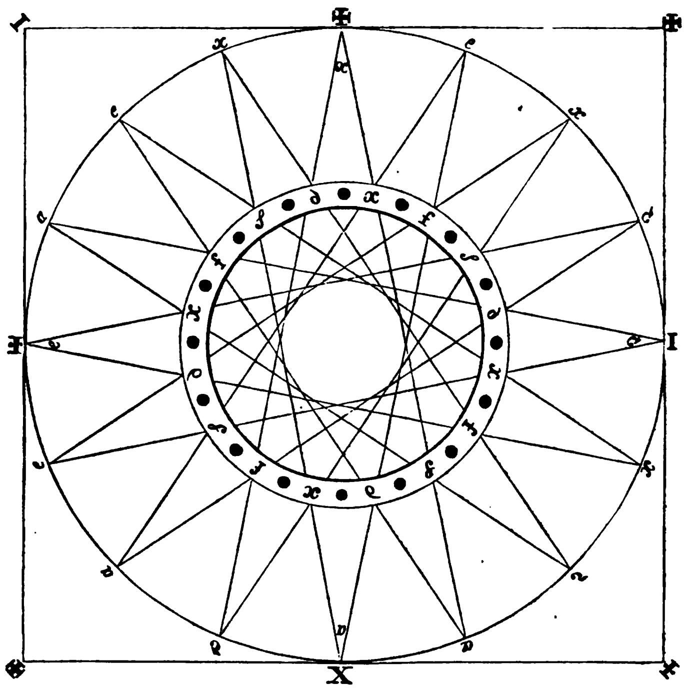
## ON THE ALGEBRAIC GAME.

I INVENTED the algebraic game about the same time that I did that theory. For when I saw some of my acquaintance, perhaps for half a day, intent on chess, wondering at their close pursuit of trifling, I asked what it was on which they were so closely occupied, and was answered, a delightful exercise of the mind. Turning this over in my thoughts, I wondered why so few applied their minds to mathematics, a pursuit at once so pleasing and so useful. Is it on account of difficulty? but many have great abilities, and decline no toil in trifles. Or is it because it is not a pleasing exercise of the mind? But what discipline or occupation whatsoever, I would ask, could better exercise skill, penetration, sagacity, every faculty of the mind? Are mathematics a game? They are no less agreeable; however, if they were to present themselves in that guise, perhaps those nice fellows, who spend their time in games, might devote themselves to this study. To this was added, the advice of that profound thinker, John Locke, on a similar occasion. I then contrived the following game as an exercise in algebra, with no great reach of mind, I allow, but such as will, I hope, easily be excused in a youth, especially one engaged in other studies.



Algebraical problems consist of given equations, which in determinate questions bring out sought quantities equal to a number. But each equation consists of two members, connected by a sign of equality, in each of which are for consideration, first, the sorts, whether they denote given or sought quantities; then, the signs by which they are connected. It is our object, then, to contrive that all these come out, to produce questions from chance, and a game, as well from the formation of the questions as from their solution. On a small board, such as is commonly used for the game of draughts, or of chess, let there be marked out a circle, inscribed in a square, and every thing set down as in the adjoining plan, except that in place of black spots there should be holes. We then shall have the table for the game. There should also be provided a slender peg of wood, which could fit in any of the holes. We now shall proceed to explain the use of these.

It may be observed that the symbols of calculation are set down at the sides and angles of the squares; moreover the sides give signs to the first, the angles to the latter members of equations. The inscribed circle is divided by sixteen points into as many equal parts, so that three points are directed to each side and angle, but some directly, some obliquely. Those which are directed obliquely to any side or angle, are common to the side and angle, but those which directly point to any side belong to no angle, but are referred equally to each adjacent one; and *vice versâ*, those which are directly pointed towards any angle belong to no side, but are to be considered referable equally to each of the adjacent ones.

In forming, then, the question, our attention should be first directed to the point which the peg marks, and the side and angle to which it belongs. These signs should be noted as those which, as we have said, connect the sorts of each member of the equation; then the peg being placed at the letter written at the said point number 1, and that being transferred to the opposite side by means of the direction of right line (as the astrologers do, assigning the reason of names by which feasts are designated) number 2. Then proceeding to the other extremity of the line, as if it were continued through the intermediate ring number 3, and so on, until the letter adjacent to the first point recurs, and so on. Hence descending by the right line to the point terminated in the convexity of the interior circle, fix the peg in either adjacent hole. The number last enumerated will show how many sought quantities, or, which is the same, how many given equations there are in the question. The former members of these are constituted by the unknown quantities taken alternately, and connected with the lateral sign, the latter, by the unknown or known quantities (as may be determined by the letter written at the internal point) connected with the sought by the angular. Moreover, *d* shows that various sorts of known quantities are to be used, *s* that only one, *f* the numeral figures 2, 3, 4, &c., *x* that the sought quantities are to be repeated. But it is to be observed, in the latter member of each equation no other unknown quantities are set down than those which are found in the first member of the following equation. What has been said will be illustrated by examples.

Let us say now, that the peg occupies the hole marked by a star, and the point to which it refers will belong to the side, the sign of which is $+$, and to the angle, the sign of which is $\times$; which signs I set down on paper, the lateral on the left, or first, then the angular. Moreover, *e* is written at the point at which I reckon 1; thence (but it is allowable of two lines to follow the direction of either) proceeding towards the left, I come on *a*, at which I reckon 2; thence, turning to *z*, I reckon 3; then, proceeding across, I meet with *e*, the letter placed at the first point, at which, reckoning 4, I proceed directly to the interior point marked by the letter *d*. There will, therefore, be four sought quantities in the question; which, connected alternately by the lateral sign +, will constitute the first members of the given equation. But the latter will be constituted of various known and unknown quantities (on account of *d*), connected by ×, the angular sign, in this way:—

$$\begin{array}{ll} a + e = yb & a = ? \\ e + y = zc & e = ? \\ y + z = ad & y = ? \\ x + a = ef & z = ? \end{array}$$

Wherefore, if we say that the peg is in the preceding hole, so that it will directly be directed to the lateral +, and we follow the direction of the left line, there will result three quantities to be investigated; and the interior point will have the letter *f*. Whence the number of the given equations, and the signs of the former members of the same, and the species of the latter are determined. But since in this case the point is circumstanced indifferently with respect to the two adjacent angles, therefore their signs are to be employed alternately; from which conditions results a question of this kind:—

$$\begin{array}{ll} a + e = 2y & a = ? \\ e + y = 3 - a & e = ? \\ y + a = 4e & y = ? \end{array}$$

But if we say that the peg is to be fixed in the following hole, the peg point should be directed towards + angular, and will equally have reference to the lateral signs + and —. Then, if you be inclined to take the right path, according to the rules laid down above, the following question will result:—

$$\begin{array}{ll} a + e = ey & a = ? \\ e - y = ay & e = ? \\ y + a = ae & y = ? \end{array}$$

* But it should first be observed, that the rules laid down admit some variety in the combinations of signs and sorts, whence it is that the point and direction being determined, various questions arise.

Secondly, that although we have already said, that a stop should be made at the recurrence of the first letter, that rule, however, can be changed at the will of each; so that we may proceed until *a, e, z, x,* successively turn up, or some of them twice; or until we reach some other limit. But we hasten to the game. First, then, let some of the players form a question for himself, according to the method just laid down. The other must do the same, then, by the same rules. So the question of each being made out, each should set himself about the solution of that which chance has given him. Then let each set down a fraction, the numerator of which is to be the number of quanti-

* See Appendix.

ties sought in his problem, and the denominator the number of degrees or equations which, whilst the question was solving, he set down on paper. He is to be the winner whose fraction is the largest.

Then, if fugitive quantities elude the eager algebraist, he must be considered to have lost all hope of victory. Nor altogether unreasonably, since it must happen rather by the fault of the selector than by mischance, that the question is undeterminate.

As often as in the sport we come to an equation adfected beyond quadratic, there will be no need of a tedious process, or construction by parabola; it is sufficient if an unknown root, its kind being changed, be regarded as known.

The solution of all the questions being finished, each should examine the work of the other, for which purpose parchment margins are convenient. As to pledges and fines, each may settle that as he pleases, for I leave these to others.

As regards the problems, they are not difficult, for otherwise they would be ill suited for amusement; but they are of that kind the solution of which will redound to the great advantage of the players whilst they strive to reach the proper path, whilst they, in their minds, run over long trains of consequences, and try to comprehend the whole series of analysis in a very brief glance. Permit me now, excellent youth, briefly to address others, for you, whom the difficulty itself attracts, have no need of an exhorter. I address you, college youths, who have energy of mind, sagacity, and penetration, but are averse from sad seclusion in the study and the lives of those generally called *Pumps*, thinking better to show your talents among your fellow idlers in play and games. You see what a mere game is algebra, and that both chance and science find place in it. Why should you not, then, come to this gaming table? For you need not here dread that which happens in cards, chess, draughts—that whilst some play others stand idly by; for whoever wish to play, can at once play and study; and some, too, make a little money. But I think I hear some one exclaiming in these words, Do you think that we can be deceived? We are not to be lured by presenting the appearance of a game, into learning a very difficult science, which must be mastered with great pains. I answer, that algebra is so far difficult as is required for a game; for if you take away all difficulty, you also take away all recreation and amusement. For all plays are so many arts and sciences. Nor is there any distinction between this and the others, but that they only regard present gratification; but from this, besides most delightful occupation, other abundant fruits are obtained. But, so far is this from having a pernicious tendency, that he is in every respect praiseworthy, according to the expression of the poet, "and has gained every point, who has combined the useful with the agreeable."

But what are those fruits which you extol? To enumerate them, mathematics in all their extent; the arts and sciences, advancing civil and military affairs, should be reviewed. For through all these is diffused the wonderful power of algebra. It is styled by all, the great, the wonderful art, the highest pinnacle of human knowledge, the kernel and key of all mathematics; and, by some, the foundation of all sciences. And, indeed, how difficult would it be to assign the limits of algebra, when it has latterly extended to natural philosophy and medicine, and daily sets about the most valuable arguments. That I may pass over other things, in the Philosophical Transactions, No. 257, there are algebraic theorems on the certainty of human evidences and traditions; and it may be laid down for certain that wherever greater and less are brought forward, wherever any ratio or proportion can be admitted, there algebra finds a place.

But perhaps some one may say, that he cares neither for mathematics, nor for any thing treated mathematically. Be it so; let us so far indulge the desire, the ignorance of persons; for I venture to maintain that this contempt proceeds from ignorance of the most exalted pursuits, and "which distinguish us from barbarians."* But is there any one who thinks slightingly of a capacious intellect, a sagacious genius, a sound judgment? If there be any one so devoid of reason, let him then disregard mathematics, the great importance of which, for forming all the best habits of the mind, is allowed by all.

Bacon somewhere, in what he has written concerning the advancement of knowledge, has observed a sort of analogy between the play of hand-ball and mathematics. To wit, as by means of that, besides the pleasure primarily aimed at, we obtain other more valuable objects, agility and strength of body, quicksightedness; so mathematical studies, besides their proper aims and uses, have that collateral one, that they abstract the mind from the senses, sharpen and confirm the talents. The ancients formerly, the wiser of the moderns now, allow this. The efficacy of algebra in rearing the intellectual powers is shown, amongst others, by Descartes, and more at length by Malbranche's Inquiry into Truth, book vi., part i., chap. 5, and part ii., chap. 8; and many other places. And those excellent rules which, in book vi., part ii., chap. 1, he lays down to be observed in the solution of questions, and which are so admirable, that an ingenious author considers that an angel could not give better; these angelic rules, I say, seem taken from algebra.

* See an Essay in English, on the Gardens of Epicurus, by Sir William Temple, Bart.

Why need I mention others, when John Locke, who, if any one did, knew well all the defects of the human intellect and their remedies, recommends, as infinitely useful, the study of mathematics in general, and of algebra especially, to all raised above the populace? See his Posthumous Works, pp. 30, 31, 32, &c., Treatise on the Conduct of the Understanding, a small and imperfect work, indeed, but which may well be preferred to the vast and elaborate volumes of others. But an author of great name thinks that mathematical pursuits require too severe meditation, and which is less suitable for a man of rank and devoted to pleasure. I answer, as Locke exhorts, that the judgment of the dissentient, St. Evremont, is set against it to no purpose. For he must be regarded an incompetent judge who, as is most probable from his life and writings, had scarcely entered the threshold. But if the bark seem hard and dry, what wonder is it? But that I may state the affair as it is, the best way is that each, making trial of the matter, follow their own judgment. Nor is there reason for raising up great difficulties because the word algebra has I know not what harsh and fearful sound. For any one can, in the short space of a month, learn the art as far as may be requisite for the game.

Having now explained our game and views, I request the mathematical reader to receive candidly these scanty first-fruits of my studies, as I will probably produce others better hereafter. For the present other studies engage, which, dry and jejune enough, have taken the place of delightful mathematics. In the meantime do you, excellent youth, accept this rhapsody of trifles as a sort of emblem of regard for yourself. Adieu.

# APPENDIX.

THAT any one may fully comprehend my purpose, I have thought it advisable to place before the view, in the following pages, all the variations of combinations and sorts in the questions which the above-stated conditions of playing admit of.

But it should be observed, in the first place, that the following formulæ, according to the modes of combination and sorts of quantities, but not all according to the number of the given equations, belong to the respective points; for often more than three quantities are to be investigated.

Secondly, that as all formulæ of questions may be had, various limits are to be laid down; otherwise only two of the four classes can belong to any one point.

I call the first point that which is directed to the lateral +, the second that next to it on the right, and so on.

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# TO THE READER.

I HAVE sometimes regretted too late to have given forth these efforts of my youth, struggling for some knowledge of mathematics only occasionally, and from my own resources. And I would still regret it, but that hence has arisen an occasion of emulation for a noble pair of geniuses, growing up as the hope of the rising generation. Nor do I boast any other claim on the republic of letters. Let these things be considered a deprecation of envy, of censure, on account of my rashness; if, indeed, I have given occasion of any.

*First Point.*

$$\begin{array}{ll} & a+e=b\times ee-bb\times yy-be\times bb-ey\times bb-y \\ & e+y=b-yy\times bb-aa\times by-bb\times ya-bb\times a \\ s & y+a=b\times aa-bb\times ee-ba\times bb-ae\times bb-e \end{array}$$

$$\begin{array}{ll} & a+e=b\times ee-bb\times yy-be\times bb-ey\times bb-y \\ & e+y=c-yy\times cc-aa\times cy-cc\times ya-cc\times a \\ d & y+a=d\times aa-dd\times ee-da\times dd-ae\times dd-e \end{array}$$

$$\begin{array}{ll} & a+e=2\times ee-22\times yy-2e\times 22-ey\times 22-y \\ & e+y=3-yy\times 33-aa\times 3y-33\times ya-33\times a \\ f & y+a=4\times aa-44\times ee-4a\times 44-ae\times 44-e \end{array}$$

$$\begin{array}{ll} & a+e=e\times ye-ye\times yy-e \\ & e+y=y-ay\times aa-ya\times y \\ x & y+a=a\times ea-ea\times ee-a \end{array}$$

*Second Point.*

$$\begin{array}{ll} & a+e=b\times eb\times y \\ & e+y=b\times yb\times a \\ s & y+a=b\times ab\times e \end{array}$$

$$\begin{array}{ll} & a+e=b\times eb\times y \\ & e+y=c\times yc\times a \\ d & y+a=d\times ad\times e \end{array}$$

$$\begin{array}{ll} & a+e=2\times e2\times y \\ & e+y=3\times y3\times a \\ f & y+a=4\times a4\times e \end{array}$$

$$\begin{array}{ll} & a+e=e\times y \\ & e+y=y\times a \\ x & y+a=a\times e \end{array}$$

*Third Point.*

$$\begin{array}{ll} & a+ea-e=e\times by\times b \\ s & e-ye+y=y\times ba\times b \\ & y+ay-a=a\times be\times b \end{array}$$

$$\begin{array}{ll} & a+ea-e=e\times by\times b \\ d & e-ye+y=y\times ca\times c \\ & y+ay-a=a\times de\times d \end{array}$$

$$\begin{array}{ll} & a+ea-e=e\times 2y\times 2 \\ f & e-ye+y=y\times 3a\times 3 \\ & y+ay-a=a\times 4e\times 4 \end{array}$$

$$\begin{array}{ll} & a+ea-e=e\times y \\ x & e-ye+y=y\times a \\ & y+ay-a=a\times e \end{array}$$

### *Fourth Point.*

$a - e = b \times e b \times y$
s $e - y = b \times y b \times a$
$y - a = b \times a b \times e$

$a - e = b \times e b \times y$
d $e - y = c \times y c \times a$
$y - a = d \times a d \times e$

$a - e = 2 \times e 2 \times y$
f $e - y = 3 \times y 3 \times a$
$y - a = 4 \times a 4 \times e$

$a - e = e \times y$
x $e - y = y \times a$
$y - a = a \times e$

### *Fifth Point.*

$a - e = e \times b b \div e y \times b b \div y b \times e e \div b b \times y y \div b$
s $e - y = y \div b b \times y a \div b b \times a b \div y y \times b b \div a a \times b$
$y - a = a \times b b \div a e \times b b \div e b \times a a \div b b \times e e \div b$

$a - e = e \times b b \div e y \times b b \div y b \times e e \div b b \times y y \div b$
d $e - y = y \div c c \times y a \div c c \div a c \div y y \times c c \div a a \times c$
$y - a = a \times d d \div a e \times d d \div e d \times a a \div d d \times e e \div d$

$a - e = e \times 2 2 \div e y \times 2 2 \div y 2 \times e e \div b 2 \times y y \div 2$
f $e - y = y \div 3 3 \times y a \div 3 3 \times a 3 \div y y \times c 3 \div a a \times 3$
$y - a = a \times 4 4 \div e o \times 4 4 \div e 4 \times a a \div d 4 \times e e \div 4$

$a - e = e \times y e \div y e \times y y \div e$
x $e - y = y \div a y \times a a \div y a \times y$
$y - a = a \times e a \div e a \times e e \div a$

### *Sixth Point.*

$a - e = b \div e b \div y e \div b y \div b$
s $e - y = b \div y b \div a y \div b a \div b$
$y - a = b \div a b \div e a \div b e \div b$

$a - e = b \div e b \div y e \div b y \div b$
d $e - y = c \div y c \div a y \div e a \div c$
$y - a = d \div a d \div e a \div d e \div d$

$a - e = 2 \div e 2 \div y e \div 2 y \div 2$
f $e - y = 3 \div y 3 \div a y \div 3 a \div 3$
$y - a = 4 \div a 4 \div e a \div 4 e \div 4$

$a - e = e \div y y \div e$
x $e - y = y \div a a \div y$
$y - a = a \div e e \div a$

### *Seventh Point.*

$a - ea \times e = e \div bb \div ey \div bb \div y$
*s* $e \times ye - y = y \div bb \div ya \div bb \div a$
$y - ay \times a = a \div bb \div ae \div bb \div e$

$a - ea \times e = e \div bb \div ey \div bb \div y$
*d* $e \times ye - y = y \div cc \div ya \div cc \div a$
$y - ay \times a = a \div dd \div ae \div dd \div e$

$a - ea \times e = e \div 22 \div ey \div 22 \div y$
*f* $e + ye - y = y \div 33 \div ya \div 33 \div a$
$y - ay \times a = a \div 44 \div ae \div 44 \div e$

$a - ea \times e = e \div yy + e$
*x* $e + ye - y = y \div aa \div y$
$y - ay \times a = a \div ee \div a$

### *Eighth Point.*

$a \times e = e \div bb \div ey \div bb \div y$
*s* $e \times y = y \div bb \div ya \div bb \div a$
$y \times a = a \div bb \div ae \div bb \div e$

$a \times e = e \div bb \div ey \div bb \div y$
*d* $e \times y = y \div cc \div ya \div cc \div a$
$y \times a = a \div dd \div ae \div dd \div e$

$a \times e = e \div 22 \div ey \div 22 \div y$
*f* $e \times y = y \div 33 \div ya \div 33 \div a$
$y \times a = a \div 44 \div ae \div 44 \div e$

$a \times e = e \div yy \div e$
*x* $e \times y = y \div aa \div y$
$y \times a = a \div ee \div a$

### *Ninth Point.*

$a \times e = b + ee \div bb + yy \div be + bb \div ey + bb \div y$
*s* $e \times y = b + yy + bb \div aa + by \div bb + ya \div bb + a$
$y \times a = b + aa \div bb + ee \div ba + bb \div ae + bb \div e$

$a \times e = b + ee \div bb + yy \div be + bb \div ey + bb \div y$
*d* $e \times y = c \div yy + cc \div aa + cy \div cc + ya \div cc + a$
$y \times a = d + aa \div dd + ee \div da + dd \div ae + dd \div e$

$a \times e = 2 + ee \div 22 + yy \div 2e + 22 \div ey + 22 \div y$
*f* $e \times y = 3 \div yy + 33 \div aa + 3y \div 33 + ya \div 33 + a$
$y \times a = 4 + aa \div 44 + ee \div 4a + 44 \div ae + 44 \div e$

$a \times e = e + ye \div ye + yy \div e$
*x* $e \times y = y \div ay + aa \div ya + y$
$y \times a = a + ea \div ea + ee \div a$

*Tenth Point.*

$$\begin{array}{ll} & a \times e = e + by + b \\ s & e \times y = y + ba + b \\ & y \times a = a + be + b \end{array}$$

$$\begin{array}{ll} & a \times e = e + by + b \\ d & e \times y = y + ca + c \\ & y \times a = a + de + d \end{array}$$

$$\begin{array}{ll} & a \times e = e + 2y + 2 \\ f & e \times y = y + 3a + 3 \\ & y \times a = a + 4e + 4 \end{array}$$

$$\begin{array}{ll} & a \times e = e + y \\ x & e \times y = y + a \\ & y \times a = a + e \end{array}$$

*Eleventh Point.*

$$\begin{array}{ll} & a \times ea \div e = e + by + b \\ s & e \div ye \times y = y + ba + b \\ & y \times ay \div a = a + be + b \end{array}$$

$$\begin{array}{ll} & a \times ea \div e = e + by + b \\ d & e \div ye \times y = y + ca + c \\ & y \times ay \div a = a + de + d \end{array}$$

$$\begin{array}{ll} & a \times ea \div e = e + 2y + 2 \\ f & e \div ye \times y = y + 3a + 3 \\ & y \times ay \div a = a + 4e + 4 \end{array}$$

$$\begin{array}{ll} & a \times ea \div e = e + y \\ x & e \div ye \times y = y + a \\ & y \times ay \div a = a + e \end{array}$$

*Twelfth Point.*

$$\begin{array}{ll} & a \div e = b + eb + y \\ s & e \div y = b + yb + a \\ & y \div a = b + ab + e \end{array}$$

$$\begin{array}{ll} & a \div e = b + eb + y \\ d & e \div y = c + yc + a \\ & y \div a = d + ad + e \end{array}$$

$$\begin{array}{ll} & a \div e = 2 + e2 + y \\ f & e \div y = 3 + y3 + a \\ & y \div a = 4 + e4 + e \end{array}$$

$$\begin{array}{ll} & a \div e = e + y \\ x & e \div y = y + a \\ & y \div a = a + e \end{array}$$

*Thirteenth Point.*

$$s\quad\begin{aligned} a \div e &= e + bb - ey + bb - yb + ee - bb + yy - b \\ e \div y &= y - bb + ya - bb + ab - yy + bb - aa + b \\ y \div a &= a + bb - ae + bb - eb + aa - bb + ee - b \end{aligned}$$

$$d\quad\begin{aligned} a \div e &= e + bb - ey + bb - yb + ee - bb + yy - b \\ e \div y &= y - cc + ya - cc + ac - yy + cc - aa + c \\ y \div a &= a + dd - ae + dd - ed + aa - dd + ee - d \end{aligned}$$

$$x\quad\begin{aligned} a \div e &= e + 22 - ey + 22 - y2 + ee - 22 + yy - 2 \\ e \div y &= y - 33 + ya - 33 + a3 - yy + 33 - aa + 3 \\ y \div a &= a + 44 - ae + 44 - e4 + aa - 44 + ee - 4 \end{aligned}$$

$$x\quad\begin{aligned} a \div e &= e + ye - ye + yy - e \\ e \div y &= y - ay + aa - ya + y \\ y \div a &= a + ea - ea + ee - a \end{aligned}$$

*Fourteenth Point.*

$$s\quad\begin{aligned} a \div e &= b - eb - ye - by - b \\ e \div y &= b - yb - ay - ba - b \\ y \div a &= b - ab - ea - be - b \end{aligned}$$

$$d\quad\begin{aligned} a \div e &= b - eb - ye - by - b \\ e \div y &= c - yc - ay - ca - c \\ y \div a &= d - ad - ea - de - d \end{aligned}$$

$$f\quad\begin{aligned} a \div e &= 2 - e2 - ye - 2y - 2 \\ e \div y &= 3 - y3 - ay - 3a - 3 \\ y \div a &= 4 - a4 - ea - 4e - 4 \end{aligned}$$

$$x\quad\begin{aligned} a \div e &= e - yy - e \\ e \div y &= y - aa - y \\ y \div a &= a - ee - a \end{aligned}$$

*Fifteenth Point.*

$$s\quad\begin{aligned} a \div ea + e &= e - by - bb - eb - y \\ e + ye \div y &= y - ba - bb - yb - a \\ y \div ay + a &= a - be - bb - ab - e \end{aligned}$$

$$d\quad\begin{aligned} a \div ea + e &= e - by - bb - eb - y \\ e + ye \div y &= y - ca - cc - yc - a \\ y \div ay + a &= a - de - dd - ad - e \end{aligned}$$

$$f\quad\begin{aligned} a \div ea + e &= e - 2y - 22 - e2 - y \\ e + ye \div y &= y - 3a - 33 - y3 - a \\ y \div ay + a &= a - 4e - 44 - a4 - a \end{aligned}$$

$$x\quad\begin{aligned} a \div ea + e &= e - yy - e \\ e + ye \div y &= y - aa - y \\ y \div ay + a &= a - ee - a \end{aligned}$$

*Sixteenth Point.*

$$s\quad\begin{array}{l} a+e=e-by-bb-eb-y \\ a+y=y-ba-bb-yb-a \\ y+a=a-be-bb-ab-e \end{array}$$

$$d\quad\begin{array}{l} a+e=e-by-bb-eb-y \\ e+y=y-ca-cc-yc-a \\ y+a=a-de-dd-ad-e \end{array}$$

$$f\quad\begin{array}{l} a+e=e-2y-22-e2-y \\ e+y=y-3a-33-y3-a \\ y+a=a-4e-44-a4-e \end{array}$$

$$x\quad\begin{array}{l} a+e=e-yy-e \\ e+y=y-aa-y \\ y+a=a-ee-a \end{array}$$

N.B. There is also another variety in the first member of the equations, where the analytic sign is found; that is, if we transpose the sorts. For instance, if in the fourth point we use $\left\{\begin{array}{l} e-a \\ y-e \\ a-y \end{array}\right\}$ in the twelfth $\left\{\begin{array}{l} e\div a \\ y\div e \\ a\div y \end{array}\right\}$ the questions will be doubled.

Lest any one should perchance suppose that in our game all possible questions are exhibited by our tables, it should be observed that they are, in fact, innumerable. For these stops can be varied without end; whence arise innumerable questions, in each of which, however, no other methods are to be followed in determining signs, combinations, and sorts, than those which are set forth in the questions alone of each odd number except unit of the quantities sought, and these are accordingly exhibited in the tables which we have given.

# CONCERNING MOTION;

OR

THE ORIGIN AND NATURE OF MOTION,

AND

THE CAUSE OF COMMUNICATING IT.

# CONCERNING MOTION.

1. It is of main import in searching for knowledge to take care that ill understood terms do not thwart us, a point which almost all philosophers inculcate, yet few attend to. Although it does not appear so difficult, especially in physical researches, which allow sensation, experiment, and geometrical reasoning. Laying aside, therefore, every prejudice originating either in usual modes of speaking, or in the authority of philosophers, we should diligently examine nature itself. Nor should the authority of any one be of such avail, that his words and expressions should be considered of value, unless they contain what is certain and clear.

2. The consideration of motion amazingly disturbed the minds of ancient philosophers, whence arose various opinions excessively difficult, not to say absurd, which, since they have now sunk into obscurity, do not deserve that we should give much attention to their discussion. But amongst recent and sounder philosophers of the present age, when motion is treated of, several words of too abstract and obscure signification occur, such as, "solicitation of gravity," "effort," "dead powers," and which diffuse obscurity over writings, in other respects of great learning, and give rise to opinions not less at variance with truth, than with the common sense of men. But it is necessary that these be discussed, not for the sake of confuting others, but on account of truth.

3. Solicitation and effort, or endeavour, in strict acceptation, are applicable merely to animated beings. When they are applied to others, they should be received in a metaphorical sense. Philosophers, however, have nothing to do with metaphors. But if we reject affection of the soul and motion of body, it will be clear to any one giving attention to the thing, that there is no distinct or plain meaning in those words.

4. As long as heavy bodies are sustained by us, we perceive in ourselves effort, fatigue, trouble; we also perceive in heavy bodies, when falling, an accelerated motion towards the centre of the earth, but nothing more, as far as our senses are concerned. However reason proves that there is some origin, or cause, of these phenomena, and this is generally called gravity. Since, however, the cause of the descent of heavy bodies is dark and unknown, gravity in that sense cannot be styled a sensible quality; consequently it is an occult quality. But we can scarcely conceive, and indeed not even scarcely, what is an

occult quality, and how any quality can act or effect any thing. It would be better then, if, putting the occult quality out of view, men would attend only to sensible effects; and abstract words, however useful they are in discussions, being omitted in speculation, the mind should be fixed on particular and concrete things, that is, on the things themselves.

5. Power in the same way is attributed to bodies, but that word is used as if it signified a known quality, distinct as well from figure, motion, and every thing sensible, as from every affection of animated life; but any person who accurately examines the subject will find that it is nothing else than an occult quality. Animal effort and corporeal motion are commonly regarded as symptoms and measures of this occult quality.

6. Thus it is plain that gravity or power is erroneously laid down as the origin of motion: for can that origin be more clearly known from its being called an occult quality? What is itself occult, explains nothing; putting out of view that the unknown acting cause can be better styled a substance than a quality. Moreover, power, gravity, and words of that kind, are employed more usually, and that not injudiciously, in the concrete, to denote the motion of bodies, the difficulty in resistance, &c.; but when they are used by philosophers to signify natures distinct and abstracted from all these, which are neither objects of sense, nor can be figured by any power of mind or imagination, they are sure to produce error and confusion.

7. But it leads many into error, that they find general and abstract words useful in disquisition, yet they cannot fully assign them a meaning. Indeed, they have been partly invented by common usage to abbreviate language, and have been partly devised by philosophers for the purposes of instruction; not that they are according to the nature of things, which indeed are singular and concrete, but because they are fit for communicating learning, because they render notions, or at least propositions, general.

8. We generally suppose that corporeal power is something easily conceived. Those who have given more attention to the subject think otherwise, as appears from the amazing obscurity of their expressions when they attempt to explain it. Torricelli says, that power and impulse are certain abstract and subtle things and quintessences, which are included in corporeal substance, as in the magic vase of Circe.* Leibnitz also, in Natura Vis Explicanda, has this passage: "Active, primitive power, which is ἡ πρώτη ἐντελέχεια, corresponds to soul or substantial form." See Transactions of the Learned Society, Lips. Thus must even the greatest men, when they give way to abstraction, have

* Matter is nothing but an enchanted vase of Circe, which serves for a receptacle of the force and the momenta of impulse. Power and impulse are such subtle abstracts, are quintessences so refined, that they cannot be enclosed in any other vessels but the inmost materiality of natural solids. See Academical Lectures.

recourse to words having no certain signification, and indeed mere scholastic shadows. We might bring forward other instances, and indeed no few of them, from the writings of recent authors, from which it is very clear, that metaphysical abstractions have not altogether given way to mechanics and experiment, but still give unnecessary trouble to philosophy.

9. From this source spring various absurdities, of which kind is this, that the force of percussion, however small, is infinitely great. Which indeed supposes that gravity is a substantial quality different from all others, and that gravitation is, as it were, an act of this quality substantially distinct from motion, but the least percussion produces a greater effect than the greatest gravitation without motion, for that causes some motion, this none. Whence it follows that the force of percussion exceeds in an infinite ratio the force of gravitation, that is, must be infinitely great. The experiments of Galileo should be consulted, and what Torricelli, Borelli, and others, have written concerning the definite force of percussion.

10. However it must be admitted, that no power can be by itself perceived, nor otherwise known, nor measured, than by its effect. But there is no effect of dead power, or simple gravity in a quiescent body, no change taking place, but there is some effect of percussion. Since, therefore, powers are proportionable to effects, we may conclude that there is no such thing as dead power. Nor still, that the power of percussion is infinite, for we ought not to regard any positive quantity as infinite, because it surpasses in an infinite ratio no quantity or nothing.

11. The force of gravity cannot be discriminated from momentum; but there is no momentum without velocity, for it is quantity of matter multiplied into velocity, and as velocity cannot be understood without motion, neither can the force of gravitation. Still further, no power can be known unless by action, and is measured by the same, but we cannot abstract the action of a body from motion; therefore, as long as a heavy body changes the figure of a piece of lead placed under it, or of a cord, so long is it moved; but, as long as it is quiescent, it does nothing, or, what is the same, is prevented from acting. Briefly, those words *dead power* and *gravitation*, although by metaphysical abstraction they are supposed to mean something different from what moves, from what is moved, from motion and rest, yet all this is nothing whatsoever.

12. If any one would say that a weight, whether hung or laid on a chord, acted on it in preventing its resuming its position by elasticity, I say that, by parity of reason, any lower body acts on the higher lying on it, because it prevents it from descending; but it can by no means be styled the action of a body, that it prevents another body to exist in the place which it occupies.

13. We occasionally feel the pressure of a gravitating body; but the annoying sensation results from the motion of that heavy body communicated to the fibres and nerves of our bodies, and changing their position, and consequently should be referred to percussion. In these things we encounter many and weighty prejudices; but they must be subdued by earnest and repeated thought, or, rather, totally extirpated.

14. To prove that any quantity is infinite, it should be shown that some finite homogeneous part is infinitely contained in it. But dead power is to the force of percussion, not as a part to a whole, but as a point to a line, according to those who maintain the infinite power of percussion. I might add much on this topic, but I fear to become prolix.

15. The principles laid down will put an end to some extraordinary disputes which have greatly engaged the attention of learned men. An example of these is that controversy concerning the proportion of powers. One side admitting that the quantity of matter being given, the momentum, motion, force, are directly as the velocity. But every one perceives that this opinion takes for granted that the force of a body is distinguished from its momentum, motion, and impulse; and that, that supposition being given up, it falls to the ground.

16. That it may appear still more clearly that a certain strange confusion has been introduced by metaphysical abstractions into the doctrine of motion, let us note how widely learned men differ in their opinions concerning power and impetus. Borelli asserts that impetus is nothing else than a degree of velocity. Some maintain that impetus and effort differ among themselves; others, that they do not differ. Some consider that the moving power is proportional to motion. Others maintain that there is some power besides the moving one, and which should be measured in a different manner, for instance, by the squares of the velocities into the quantities of matter. But there is no end in pursuing these things.

17. Force, gravity, attraction, and words of this sort, are serviceable for reasonings and computations concerning motion and bodies in motion, but not for understanding the simple nature of motion itself, or for denoting so many distinct qualities. Certainly, as far as regards attraction, it is clear that it is adopted by Newton, not as a real, physical quality, but merely as a mathematical hypothesis. Still further, Leibnitz, distinguishing elementary effort, or solicitation, from impetus, allows that those things are not in reality found in nature, but produced by abstraction.

18. Such also are the composition and resolution of any direct forces into any oblique ones, by the diagonal and sides of a parallelogram. These are serviceable for mechanics and computation; but it is one thing to be serviceable to computation

and mathematics, and another to set forth the nature of things.

19. Of the moderns there are many of opinion, that motion is not destroyed nor produced anew, but that there is always the same quantity of motion. Aristotle, also, proposed that question, whether motion be created and corrupt, or from eternity. That sensible motion perishes is plain from our senses; but they will have it that the same impetus, effort, or the same sum of forces, remains. Whence Borelli maintains, that force is not lessened in percussion, but expanded; also that contrary impulses are received and preserved in the same body. Leibnitz also maintains that effort is every where and always in matter. It must be allowed that these things are too abstract and obscure, and of the same sort as substantial forms and entelechiæ.

20. Those who, to explain the cause and origin of motion, make use either of the hylarchic principle, or the need of nature, or appetite, or, finally, natural instinct, should be deemed rather to have said something than to have thought any thing. Nor is there much difference between such persons and those* who suppose that the parts of the earth move themselves, or that spirits are implanted in them in the same way as are forms, and in this way assign the cause of the acceleration of heavy bodies falling; or he† who maintained that in bodies, besides solid extension, there should also be allowed something whence the consideration of forces might arise. For all these either lay down nothing particular and determinate, or, if there be any thing in what they say, it will be as difficult to explain as that very thing on account of explaining which it is brought forward.

21. It is to no purpose for explaining nature, to bring forward what is neither open to the senses nor can be understood by the reason. We should consider, therefore, what the senses, what experience, what reason, resting on them, impresses on us. There are two chief sorts of things—body and mind. By the aid of our senses we perceive something extended, solid, moveable, having figure and other qualities, the objects of our senses; and, by some internal consciousness, we know of something sentient, perceptive, and intelligent. Moreover, we perceive that these things differ widely from each other, and are thoroughly heterogeneous. I am speaking of things that are known; we need say nothing of things unknown.

22. That which we know and call body, in no respect contains any thing in itself which can be the origin or efficient cause of motion; for impenetrability, extension, figure, include or denote no power of producing motion; nay, on the contrary, examining singly not only those but other qualities of bodies, we shall find all their qualities passive, and that there is nothing in them active, and which can in any way be regarded as the source and origin of motion. As to gravity, we have shown

* Borelli. † Leibnitz.

that word to signify nothing known or distinct from the sensible effect the cause of which is the object of inquiry; and, indeed, when we call a body heavy, we understand nothing more than that it tends downwards, not regarding the cause of this sensible effect.

23. We can, therefore, without hesitation, state respecting body, that it is not the origin of motion. Wherefore, if any one maintains, that, in addition to solid extension and its modifications, the word body implies occult virtue, form, essence, he may with vain toil dispute without notions, and indulge in an abuse of words which express nothing distinctly. But the wiser course of philosophers would have been, to have abstained altogether from abstract and general notions; if, indeed, notions which cannot be understood ought to be expressed at all.

24. We know what is contained in the idea of body; but what we know in body it is clear is not the origin of motion. But those who attribute to body something unknown, of which they have no idea, and which they call the origin of motion: such persons say nothing more than that the origin of motion is unknown. But we need no longer dwell on such subtleties.

25. Besides corporeal beings, there is another class, that of thinking beings. That these have a power of moving bodies, we know by our own experience; since our minds can at pleasure excite and stop the movements of our limbs, however it is effected. This is certain, that our bodies are moved at the will of our minds, which consequently may not improperly be styled the origin of motion; a particular and subordinate one, indeed, and which itself depends on the first universal origin.

26. Heavy bodies tend downwards, although agitated by no apparent impulse. We must not, however, therefore suppose that the origin of motion resides in them; on which fact Aristotle thus reasons:—Heavy and light things, he observes, are not moved of themselves; for that would be vitality, and they could stop themselves. All heavy bodies tend to the centre of the earth, by a certain and constant law; nor is there perceived any principle or power of stopping or diminishing that motion, or of increasing it, except by a fixed proportion, or of in any way modifying it; consequently, their condition is merely passive. Moreover, the same thing should, strictly and accurately speaking, be said respecting percussive bodies. Those bodies, as long as they are moved, and also in the very moment of percussion, are as much passive as when they are at rest. A body at rest acts as much as a body in motion; as Newton admits, when he says that the force of *inertia* is the same with *impetus*. But an inert body does nothing; so neither does a body moved.

27. In reality, a body equally persists in each state, either of rest or of motion. But its doing so can no more be called an

action of body, than its existence can be called its action. Its persevering is nothing more than a continuation in the same mode of existence, which cannot properly be called .action. But the resistance which we experience in stopping a body in motion, we make out to be an action of it; but this is a delusion. For, in reality, that resistance which we perceive is an impression in ourselves; nor does it prove that body acts, but that we have an impression; and it is plain that we should have the same impression whether the body were moved by itself, or were impelled by some other principle.

28. Action and reaction are said to be in bodies; and such expressions are convenient for mechanical demonstrations. But we should be on our guard not therefore to suppose in them some real virtue which may be the cause or origin of motion. For those words are to be understood in the same way as the word attraction; and as this last is merely a mathematical hypothesis, and not a physical quality, the same should be understood concerning those, and for the same reason. For as truth, and the use of theorems concerning the mutual attraction of bodies remain unshaken in mechanical philosophy, as founded on the motion of bodies, whether that motion may be supposed by the action of bodies mutually attracting each other, or by the action of some agent different from body, impelling and stopping bodies; for the same reason, whatsoever things have been laid down concerning the rules and laws of motion, and the theorems deduced from them, remain unquestionable, provided the sensible effects and reasonings depending on them be admitted.

29. Let extension, solidity, figure, be taken away from the idea of body, nothing will remain; but these qualities are indifferent to motion, nor have they any thing in them which can be styled the origin. This is clear from our ideas themselves. If, therefore, by the word body be signified that which we conceive, it is quite clear that the origin of motion cannot be implied, for no part or attribute of it is a real, efficient cause, which can produce motion. But to use a word, and attach no idea to it, is in truth unworthy of a philosopher.

30. We find that there is a thinking, active being, which we learn, from experience, to be the origin of motion, in us. This we style soul, mind, spirit. We find that there is also a being extended, inert, moveable; which differs altogether from the other, and constitutes a new class. Anaxagoras, a man of great sagacity, who first perceived the difference between thinking and extended being, asserted that mind had nothing in common with body, as appears from the first book of Aristotle on the Mind. Among the moderns, Descartes also, has very well laid this down. Some, after him, have made this plain truth confused and difficult by their obscure expressions.

31. It is clear from what we have laid down, that those who affirm that active power, action, the origin of motion, are actually in body, maintain an opinion unsupported by experience; that they prop it up by obscure and general terms, nor do they completely understand themselves. They, on the contrary, who maintain that mind is the origin of motion, express an opinion supported by their own experience, and confirmed by the opinion of the most learned men in all ages.

32. Anaxagoras first had recourse to mind (τὸν νοῦν), as that which impressed motion on inert matter. Which opinion Aristotle also maintains and confirms by many arguments, justly asserting that the first mover is immoveable, indivisible, and has no magnitude. To say that every thing producing motion must be moveable, he rightly maintains, is as if a person would maintain that every thing which builds must be capable of being built, Physics, lib. viii. Plato also, in Timæus, lays it down, that this material machine, or the visible world, is moved and animated by a mind which evades all sensation. Still further, the Cartesians of the present day acknowledge God as the origin of motion. Newton, also, every where intimates, by no means obscurely, that motion not only originally proceeds from the Deity, but that still the mundane system is kept in motion by his power. This agrees with scripture, and is confirmed by the calculations of the learned. For although the Peripatetics lay it down that nature is the origin of motion and rest, they interpret it to be the Deity acting as motion. For they understand that all the bodies of this mundane system are moved by an all-powerful mind, according to a certain and constant reason.

33. But those who attribute a vital principle to bodies, devise something obscure and ill agreeing with reality. For what else is being endowed with vital principle than to live; or to live, than to move itself, stop, and change its state? Now the philosophers of the present day lay it down as an indisputable principle, that every body perseveres in its state, either of rest or of uniform rectilinear motion, unless so far as it is from some external cause compelled to change that state; on the contrary, in mind, we perceive a power of changing its own state, as well as that of other beings, which is called vital, and fully distinguishes mind from body.

34. The moderns consider motion and rest in bodies as two states of existence, in each of which every body naturally would remain inert, unless external force impelled it. Whence we may say that the cause of motion and rest is the same as that of the existence of body, for it does not seem that we should look out for any cause of the successive existence of body in different parts of space, than that whence results

the successive existence of body in different parts of time. But to treat of the God Almighty and All Good, the Creator and Preserver of all things, and in what manner all things depend on the supreme and true Being, although the most exalted branch of human learning, belongs rather to primary philosophy, or metaphysics and theology, than to natural philosophy, which at present is almost completely restricted to experiments and mechanics. Therefore natural philosophy either presupposes a knowledge of the Deity, or derives it from some science of a higher order. Although it is most true that the investigation of nature every where affords excellent arguments to the higher sciences, for illustrating and proving the wisdom, goodness, and power of God.

35. This not being sufficiently understood is the reason why some unadvisedly regret the mathematical principles of physics on that ground, that they do not assign efficient causes of things; when, in truth, it appertains to physics or mechanics to state the rules only of impulse and attraction, and not the efficient causes, in a word, the laws of motion; and from these, when received, to assign the solution of particular phenomena, but not their efficient cause.

36. It will be of great use to consider what origin properly is, and in what sense it must be taken amongst philosophers. Now the true, efficient, and preserving cause of all things is most properly called their source and origin. But the word *principia*, when applied to experimental philosophy, properly signifies the grounds on which it rests, or the sources from which is derived (I do not say the existence, but) the knowledge of material beings, these grounds being sensation and experience. In the same way, in mechanical philosophy, we mean by *principia* that constituting the grounds and extent of the whole science; being those primary laws of motion which, confirmed by experiment, are cultivated and rendered universal by reason. These laws of motion are appropriately called *principia*, principles, since from them are derived as well the general theorems of mechanics as the particular explanations of phenomena.

37. Then truly something can be said to be explained mechanically when it is reduced to those most simple and universal principles, and is by accurate reasoning shown to be suitable and connected with them. For the laws of nature being once ascertained, it remains for the philosopher to show that each thing necessarily follows in conformity with these laws; that is, that every phenomenon necessarily results from the principles. This is to explain and solve the phenomena; that is to assign the reason why they take place.

38. The human mind delights in extending and enlarging its knowledge. But for this purpose general notions and proposi-

tions must be formed, in which, in some way or other, are comprised particular propositions and facts; which are then considered to be understood when they are deduced from them by continued consequence. This is well known to geometricians. In mechanics, also, the course is, first, to lay down notions; that is, definitions and elementary and general positions concerning motion; from which subsequently, in the mathematical style, more remote and less general conclusions are drawn. And, as by the application of geometrical theorems the magnitudes of particular bodies are measured, so by the application of the general theorems of mechanics we ascertain and determine the motions of any parts of the mundane system, and the phenomena depending on them; and this the investigator of physics should exclusively aim at.

39. And as geometricians, for the sake of practice, devise many things which they neither themselves can contrive nor find in the nature of things, for the same reason those who treat of mechanics employ certain abstract and general words, and assume power, action, attraction, solicitation, &c.; which are of the first utility for theories, enunciations, and computations concerning motion, although in actual truth and bodies really they are sought in vain, as much as are those things imagined by mathematical abstraction.

40. In reality, by the use of our senses we perceive nothing except effects or sensible qualities, and material beings in all respects passive, whether at rest or in motion; and reason and experience indicate nothing active except mind or soul; whatever is imagined more than this must be regarded of the same sort as those mathematical hypotheses and abstractions; and we should thoroughly bear this in mind. Unless this take place we may easily relapse into the obscure subtlety of the schoolmen, which for so many ages infected philosophy like a dreadful plague.

41. The mechanical principles and universal laws of motion, or of nature, happily discovered in the last century, and treated of and applied by the aid of geometry, have thrown a wonderful light on philosophy. But the metaphysical principles, and the real efficient causes of motion, and of the existence of bodies, and the attributes of bodies, by no means belong to mechanics or experiments; nor can they throw light on them, except this far, that known previously, they may serve to define the limits of physics, and thus to do away with difficulties and questions which are foreign to them.

42. Those who derive the origin of motion from spirit, understand by the name either a corporeal or incorporeal being. If they understand a corporeal being, however subtle, the difficulty recurs; but if an incorporeal, however true may be their opinion,

it does not properly belong to physics. Wherefore, if any extend natural philosophy beyond the limits of experiments and mechanics, so that it embraces the knowledge of immaterial, unextended things, the wider extent of the word admits the treating of soul and mind, or the vital principle. But it will be more commodious, according to the usage now generally received, to distinguish between sciences; so that each be circumscribed by proper limits, and the natural philosopher be entirely engaged in experiments on the laws of motion, and in mechanical principles, and the reasonings deduced from them; but what he may bring forward concerning other things he should consider the offspring of a more exalted science. For from the knowledge of the laws of nature the most beautiful theories, as well as mechanical processes useful to life, proceed; but from knowledge concerning the Author of nature himself arise speculations unquestionably of the highest order, but metaphysical, moral, and theological.

43. So far concerning the principles, we must now treat of the nature, of motion. And it, indeed, since it is clearly perceived by the senses, has been obscured not so much by its own nature, as by the learned conjectures of philosophers. Motion never is presented to our senses without material mass, space, and time. There are some, however, who try to contemplate motion as a certain simple and abstract idea, and abstracted from all other things. But that very subtle and fine-drawn idea evades the acuteness of our intellects, as any one can experience by meditation. Hence result great difficulties concerning the nature of motion, and definitions more obscure than the thing which they are intended to explain. Such is that of Aristotle and the schoolmen, who say that motion is the act of what is moveable, as far as it is moveable; or the acting of a being in power as far as it is in power. Such is that also of a celebrated author of later times, who asserts that there is nothing real in motion except that momentary thing which ought to be comprised in power struggling for a change. Moreover, it is plain that the authors of these and of similar definitions have bent their minds on explaining the abstract nature of motion, laying aside all consideration of time and place; but how that abstract quintessence of motion, as I may call it, can be understood, I cannot perceive.

44. Nor are they content with this; but they go further, and separate and discriminate from each other the parts of motion, the distinct ideas of which they attempt to form, as of beings really distinct; for there are some who distinguish if motion from movement, regarding the former as the instantaneous element of movement. Still further; they regard velocity, effort, force, impetus, as many things differing essen-

tially, each of which is presented to the intellect by a peculiar idea, distinct and abstracted from all others. But if what we have before treated of be admitted, we need spend no more time in these discussions.

45. Many also define motion by means of transition, not recollecting that transition itself cannot be understood without motion, and ought to be defined by means of motion. So true is it, that definitions which throw light on some things, cause darkness in others. And, indeed, whatever we perceive by our senses, scarcely any one can make better known or more clear by definitions. Allured by the vain hope of which, philosophers have made the easiest things most difficult; and have entangled their minds in difficulties which they have themselves, for the most part, caused. From this desire of defining, as well as of abstracting, many questions at once both subtle and useless, which have arisen both concerning motion and other things, have fruitlessly racked men's minds; so that Aristotle freely, on many occasions, confesses that motion is some act difficult to be known; and some of the ancients became so hackneyed in these trifles that they altogether denied the existence of motion.

46. But it is grievous to be detained by trifles of this kind. Let it suffice to show the sources of the solutions, and to add, that all the paradoxes and thorny theories (such, for instance, as those which treat of infinites) which have been introduced into mathematics, concerning the unlimited division of time and space, have been introduced into the definitions concerning motion; but all things of that kind motion has in common with time and space, or rather refers to them.

47. Still further, as the too great division or abstraction of things in reality inseparable, so on the other hand the combination, or rather confusion, of things most different, has rendered the nature of motion perplexed. For it is usual to confound motion with the efficient cause of motion. Whence motion is as it were twofold; having one face obvious to our senses, the other wrapped in dark night. Hence result obscurity, confusion, and various paradoxes concerning motion. Whilst that is rashly attributed to effect, which in reality can belong only to cause.

48. Hence arises the opinion, that the same quantity of motion is always kept up, which any one must know to be false, unless it be understood concerning the force or power of the cause, whether that cause be termed nature or νοῦς, or whatever sort of agent it may be. Aristotle, in the eighth book of his Physics, where he inquires whether motion be created and corrupt, or be in all things from eternity as an immortal life, seems to have understood rather a vital principle, than an external effect or change of place.

49. Hence it is that many think that motion is not a merely passive quality of body. If this mean what is presented to our senses in the case of motion, no one can doubt that it is altogether passive; for what has the successive existence of body in various places that has any thing to do with action, or can be any thing but mere inert effect?

50. The Peripatetics, who maintain that motion is one act of two beings, of the mover and of the moved, do not sufficiently distinguish cause from effect. In the same way, those who imagine endeavour or effort in motion, or think that the same body is impelled towards different parts, seem to sport with the same confusion of ideas, the same ambiguity of words.

51. It is of great use, as in all other things, so in science, to investigate accurately about motion, as well for understanding the opinions of others, as for accurately enunciating our own; and unless there be some fault in this respect, I scarcely think that a dispute can be raised, whether body be indifferent to motion and rest, or not. For since it is clear from experience, that it is a primary law of nature that a body should persevere in a state of motion or of rest as long as nothing occurs from another quarter to change that state, and therefore it may be concluded that the inert quality is in different respects either resistance or impetus: in this sense certainly body can be said to be indifferent as to rest or motion. For it is as difficult to cause rest in a moving body, as to cause motion in a body at rest; but when a body equally preserves either state, why should it not be said to be indifferently circumstanced as to both?

52. But the Peripatetics, according to the variety of the changes which any body can undergo, distinguished various kinds of motion. Those who at present treat the subject take into account only local motion. But local motion cannot be understood, unless we also understand the meaning of *place*, which is by the moderns defined to be the part of space which body occupies, and therefore in reference to space it is divided into absolute and relative. For they distinguish between absolute and true space, and that which is apparent or relative. They maintain indeed that there exists in every direction an immense immoveable space, not the object of sensation, but pervading and embracing all bodies, and this they call absolute space. But the space comprehended or marked out by body, and so subjected to our senses, is called relative, apparent, common space.

53. Let us then imagine all bodies to be destroyed and annihilated. What then remains they call absolute space, all relation resulting from the situation and distances of bodies, as well as the bodies themselves, being done away with. Now this space

is infinite, immoveable, indivisible, not the object of sensation, without relation and without distinction. That is, all its attributes are privative or negative; therefore it seems to be a mere nothing. The only difficulty results from its being extended, for extension is a positive quality. But what sort of extension is that, which can be neither divided nor measured, no part of which we can either perceive by our senses, or figure in the imagination? for nothing can enter the imagination which from the nature of the thing is not possible to be perceived by sensation, since imagination is nothing else than a faculty representing the objects of sensation, either existing in act, or at least being possible. It also evades pure intellect, since that faculty is only conversant about spiritual and unextended things, such as our minds, their habits, passions, virtues, and such things. Let us, then, take away mere words from absolute space, and nothing remains in sensation, imagination, or intellect; nothing, therefore, is denoted by them but mere privation or negation, that is, mere nothing.

54. It must be allowed, that with respect to this subject, we are entangled with the greatest prejudices, to free ourselves from which we must exert the whole vigour of our minds; for many are so far from considering absolute space as a non-entity, that they think it the only thing, except God, which cannot be annihilated; and maintain that it necessarily exists by its own nature, and is eternal and uncreated, and consequently participates in the divine attributes. But since it is most certain that every thing which we denote by names is in some respect known by qualities and relations (for it would be silly to make use of words having no known thing, no notion, no idea or conception attached), let us diligently inquire whether we can form any idea of that pure, real, absolute space, which would continue to exist after the annihilation of all body. Examining such an idea accurately, I find it to be the most perfect idea of nothing, if it can be called an idea. This has been the case with me after I have used all my diligence; and I think that others will find the same if they use the same diligence.

55. It sometimes is wont to lead us astray, that all other bodies being in imagination done away with, we suppose that our own bodies remain; and supposing this, we imagine the freest motion of our limbs in every direction. But motion cannot be conceived without space. Still if we consider the matter more attentively, it is clear, 1st, that we conceive relative space marked out by the parts of our body; 2nd, a free power of moving our limbs, checked by nothing, and nothing besides this. However, we erroneously suppose that some other thing, called infinite space, really exists, which allows us free power of moving our bodies; for nothing more than the absence of other

bodies is required for this. Which absence or privation of body we must allow to be nothing positive.*

56. But unless these things be freely and closely examined, words and sounds are but of little avail. Now it will be clear to any one attentively considering, that whatever is predicated of pure space may be predicated of nothing. By which means the human mind is easily freed from great difficulties; and, at the same time, of attributing necessary existence to any being except the almighty and all-good God alone.

57. It were easy to confirm our opinion by arguments derived *à posteriori*, as it is called, by proposing questions concerning absolute space for instance, whether it be a substance or an accident, whether created or uncreated, and pointing out the absurdities resulting from either position. But I must be brief. It, however, ought not to be omitted, that Democritus formerly confirmed this opinion by his support, as Aristotle mentions in Physics, lib. i., where he has this passage, "Democritus lays down as principles, solidity and void, of which he says, that the one is as that which is, the other is as that which is not. But if any one would raise a doubt, that the distinction between absolute and relative space is admitted by great philosophers, and that many famous theorems are founded on that ground, it will appear from what follows, that such a ground is vain.

58. From what has been laid down, it is plain that it is not consistent that we should define the true place of a body to be that part of absolute space which the body occupies, and true or absolute motion to be the change of true and absolute place; since all place is relative, as well as all motion. But that this may appear more clearly it should be observed, that no motion can be understood without some direction or determination, which, indeed, cannot be understood unless, besides the body that is moved, our bodies also, or some other thing, be understood to exist: for upwards, downwards, to the left, to the right, and all places and regions, are founded in something relative, and necessarily imply some body distinct from that which is moved; so that if all other bodies were annihilated, and only one, a globe for instance, were supposed to exist, no motion could be conceived in it: so necessary is it that some body be given, by the situation of which motion can be determined. The truth of that opinion will most clearly appear, if we form a clear notion of the annihilation of all bodies, as well our own as others, except that globe.

59. Moreover, let there be two globes imagined, and nothing corporeal besides. Let it then be imagined that forces are in some way applied: whatsoever we may understand by the appli-

* See what has been urged against absolute space in the book concerning the Principles of Human Knowledge. Vol. I. p. 131.

cation of forces, a circular motion of the two globes about a common centre cannot be conceived in imagination. Let us suppose then, the heavens of fixed stars to be created immediately; motion will be conceived from the conceived course of the globes to different parts of that heaven. For since motion is in its nature relative, it cannot be conceived before that correlative bodies be given, as also no other relation can be conceived without correlatives.

60. As to circular motion, many think that true circular motion increasing, a body tends more and more to recede from the axis. But this results from that circumstance, because, since circular motion can be regarded as every moment resulting from two directions, one that of the radius, the other that of the tangent, if the force be increased in this last direction, then the body in motion will recede from the centre, but the orbit will cease to be circular. But if the forces be equally increased in both directions, the circular motion will continue, but accelerated; which will prove that the forces, both of receding from the axis and of approaching it, are increased. It must therefore be said that water, whirled round in a bucket, mounts up the side of the vessel, because, fresh forces being applied to each particle of water in the direction of the tangent in the same instant, fresh centripetal forces are not applied. From which experiment it by no means follows, that the absolute circular motion is necessarily indicated by the forces of the motion receding from the axis. Moreover, how we should understand the words forces and efforts of bodies appears plainly from what has been already written.

61. In the same way as a curve may be considered as consisting of an infinite number of right lines, although not in reality consisting of them, but because the hypothesis is serviceable for geometry, so a circular motion may be regarded as resulting from an infinity of rectilineal directions, an hypothesis useful in mechanical philosophy. It is not, however, to be maintained, that it is impossible that the centre of gravity of each body may successively exist in each point of the periphery of a circle, no account being taken of any rectilineal direction, either in the tangent or in the radius.

62. It should not be omitted, that the motion of a stone in a sling, or of water whirled round in a bucket, cannot be called a really circular motion, according to the notion of those who define the true limits of body by the parts of absolute space, since it is wonderfully compounded of the motions, not only of the sling or bucket, but of the diurnal motion of the earth round its axis, its monthly motion round the common centre of gravity of the earth and moon, and its annual round the sun; on account of which every particle of the stone or water describes a line

very widely differing from a circular one. Neither, in reality, is that, as it is supposed, an axifugal tendency, since it does not regard any one axis with reference to absolute space, if the existence of absolute space be admitted; so I do not see how it can be called a single tendency, to which a motion, truly circular, corresponds as to its proper and adequate effect.

63. No motion can be perceived or measured except by what is the object of sensation. Since, therefore, absolute space is not obvious to our senses, it must be of no avail for distinguishing. Besides, determination or direction is essential to motion; but that consists in relation; therefore it is impossible to conceive absolute motion.

64. Moreover, since according to the difference of relative place, the motion of the same body may be various, and indeed a body may be said to be moved in one respect, and motionless in another; for determining true motion and true rest, for the purpose of removing ambiguity and advancing the mechanical philosophy of those who contemplate the system of things in a more enlarged way, it will be sufficient, instead of absolute space, to regard relative space as marked out by the heavens of the fixed stars considered as at rest. But motion and rest, marked out by such relative space, can be conveniently used instead of the absolute, which can by no distinction be discriminated from them. For howsoever forces may be impressed, whatever tendencies there may be, let us admit that motion is distinguished by means of action on bodies, it will, however, never follow from thence, that there are absolute space and place, and that its change is the true place.

65. The laws and effects of motion, and theorems containing their proportions and calculations, according to their different courses, also their accelerations and different directions, and mediums of more or less resistance; all these come out without the calculation of absolute motion. As is plain from this, that according to the principles of those who bring forward absolute motion, it cannot be known by any symptom, whether the whole frame of things rests or is moved uniformly in direction; it is clear that the absolute motion of no body can be known.

66. From what has been said, it appears, that to ascertain the true nature of motion, it will be of great avail, 1st, to distinguish between mathematical hypotheses, and the nature of things; 2nd, to beware of abstractions; 3rd, to consider motion as something, the object of sensation, or at least of imagination; and to be content with relative measures: which if we do, at the same time the finest theorems of mechanical philosophy, by means of which the recesses of nature are disclosed, and the system of the world subjected to human calculation, will remain uninjured, and the consideration of motion freed from a thousand minute sub-

tleties and abstractions. And let it suffice to say so much concerning the nature of motion.

67. It remains that we should treat of the cause of the communication of motion. But most consider, that force impressed on a moveable body is the cause of motion in it. Nevertheless it results from what has been laid down, that they do not assign a known cause of motion, and distinct from body and motion. It is further clear, that force is not a thing certain and determinate, from the circumstance, that men of the greatest powers of mind advance different, and even contrary things, though retaining truth in the consequences. For Newton says, that force impressed consists in action alone, and is an action exercised on body to change its state, nor that it continues after. Torricelli contends that a certain accumulation or aggregation of forces impressed by percussion, is received into the moved body and remains there, and constitutes the impetus; Borelli and some others maintain the same. But although Newton and Torricelli seem to differ, each advancing things consistent with themselves, the matter is well enough explained by both. For forces attributed to bodies are as much mathematical hypotheses as attractive powers assigned to the planets and sun. Mathematical things, however, have no stable essence in nature, but depend on the notion of the definer, whence the same thing can be differently explained.

68. Let us lay it down, that the new motion is preserved in the struck body, either by innate force, by which any body continues in its state of motion or rest, uniformly in direction; or by force impressed, and during the impression received into the struck body, and there remaining; it will be the same as to reality, the difference existing only in name. In like manner, when the striking body loses, and the struck acquires motion, it is little to the purpose to dispute whether the motion acquired be numerically the same with that lost; for it leads us into metaphysical and verbal disputes concerning identity; therefore, whether we say that motion passes from the striking body into the struck, or that motion is generated anew in the struck body and destroyed in the striking, it amounts to the same thing. In each instance it is meant that one body acquires motion and another loses it, and nothing more.

69. That the intelligence which moves and embraces this whole corporeal mass, and is the efficient cause of motion, is strictly and properly speaking the cause of the communication of the same, I will not deny. However, in physical philosophy we ought to deduce the causes and solutions of phenomena from mechanical principles. The thing is explained physically, not by assigning the really acting and immaterial cause, but by demonstrating its connexion with mechanical principles. Of that kind is that, that

action and reaction are always contrary and equal, from which, as from a primary principle and source, are deduced the rules concerning the communication of motion, which have been ascertained and demonstrated by moderns to the great benefit of science.

70. Let it suffice us to hint that that principle can be declared in another way. For if the true nature of things, rather than abstract mathematics, be regarded, it will seem to be more properly said, that in attraction or percussion, the passive rather than the active quality of bodies is equal. For instance, a stone tied by a rope to a horse is as much drawn towards the horse as the horse towards the stone. A moved body also dashed against another at rest suffers the same change with the quiescent body, and as to the real effect the striking body is also struck, the struck body striking. But the change in each instance, as well in the body of the horse as in the stone, in the body in motion and at rest, is a merely passive state. But it does not appear that there is a force, a virtue, or material action, really and properly causing such effects. A body in motion is dashed against one at rest; but we use an active mode of expression, saying, that the one impels the other, and not improperly in mechanics, where the mathematical rather than the actual causes of things are considered.

71. In physics, sensation and experience, which only reach apparent effects, are admitted; in mechanics, the abstract notions of mathematicians are admitted. In primary philosophy, or metaphysics, we treat of immaterial things, causes, truth, and the existence of things. The writer on physics contemplates the series or succession of the objects of sense, by what laws they are connected, and in what order; observing what precedes as a cause, what follows as an effect. And in this way we say that a moved body is the cause of motion in another, or impresses motion on it; also that it draws or impels. In which sense secondary corporeal causes ought to be understood, no account being taken of the actual place of the forces, or active powers, or of the real cause in which they are. Moreover, beyond body, figure, motion, the primary axioms of mechanical science can be styled cause, or mechanical principles, being regarded as the causes of what follow them.

72. The truly active causes can be extracted only by meditation and reasoning from the shades in which they are involved, and thus at all become known. But it is the province of primary philosophy, or of metaphysics, to treat of them. Wherefore if its own province were assigned to each science, its limits marked out, its principles and objects accurately distinguished, we could treat of what belongs to each with greater facility and perspicuity.

# THE ANALYST:

OR

## A DISCOURSE ADDRESSED TO AN INFIDEL MATHEMATICIAN:

WHEREIN IT IS EXAMINED

WHETHER THE OBJECT, PRINCIPLES, AND INFERENCES OF THE MODERN ANALYSIS ARE MORE DISTINCTLY CONCEIVED, OR MORE EVIDENTLY DEDUCED, THAN RELIGIOUS MYSTERIES AND POINTS OF FAITH.

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# THE ANALYST.

I. Though I am a stranger to your person, yet I am not, Sir, a stranger to the reputation you have acquired in that branch of learning which hath been your peculiar study; nor to the authority that you therefore assume in things foreign to your profession; nor to the abuse that you, and too many more of the like character, are known to make of such undue authority, to the misleading of unwary persons in matters of the highest concernment, and whereof your mathematical knowledge can by no means qualify you to be a competent judge. Equity indeed and good sense would incline one to disregard the judgment of men, in points which they have not considered or examined. But several who make the loudest claim to those qualities do nevertheless the very thing they would seem to despise, clothing themselves in the livery of other men's opinions, and putting on a general deference for the judgment of you, gentlemen, who are presumed to be of all men the greatest masters of reason, to be most conversant about distinct ideas, and never to take things upon trust, but always clearly to see your way, as men whose constant employment is the deducing truth by the justest inference from the most evident principles. With this bias on their minds, they submit to your decisions where you have no right to decide. And that this is one short way of making infidels, I am credibly informed.

II. Whereas then it is supposed, that you apprehend more distinctly, consider more closely, infer more justly, conclude more accurately than other men, and that you are therefore less religious because more judicious, I shall claim the privilege of a free-thinker; and take the liberty to inquire into the object, principles, and method of demonstration admitted by the mathematicians of the present age, with the same freedom that you presume to treat the principles and mysteries of religion; to the end that all men may see what right you have to lead, or what encouragement others have to follow you. It hath been an old remark, that geometry is an excellent logic. And it must be owned, that when the definitions are clear; when the postulata cannot be refused, nor the axioms denied; when from the distinct contemplation and comparison of figures, their properties are derived, by a perpetual well-connected chain of consequences, the objects being still kept in view, and the attention ever fixed upon them; there is acquired a habit of reasoning, close and exact and methodical: which habit strengthens and sharpens the mind, and being transferred to other subjects, is of general use in the inquiry after truth. But how far this is the case of our geometrical analysts, it may be worth while to consider.

III. The method of fluxions is the general key, by help whereof the modern mathematicians unlock the secrets of geometry, and consequently of nature. And as it is that which hath enabled them so remarkably to outgo the ancients in discovering theorems and solving problems, the exercise and application thereof is become the main, if not sole, employment of all those who in this age pass for profound geometers. But whether this method be clear or obscure, consistent or repugnant, demonstrative or precarious, as I shall inquire with the utmost impartiality, so I submit my inquiry to your own judgment, and that of every candid reader. Lines are supposed to be generated* by the motion of points, planes by the motion of lines, and solids by the motion of planes. And whereas quantities generated in equal times are greater or lesser according to the greater or lesser velocity wherewith they increase and are generated, a method hath been found to determine quantities from the velocities of their generating motions. And such velocities are called fluxions: and the quantities generated are called flowing quantities. These fluxions are said to be nearly as the increments of the flowing quantities, generated in the least equal particles of time; and to be accurately in the first proportion of the nascent, or in the last of the evanescent increments. Sometimes, instead of velocities, the momentaneous increments or decrements of undetermined flowing quantities are considered, under the appellation of moments.

IV. By moments we are not to understand finite particles. These are said not to be moments, but quantities generated from moments, which last are only the nascent principles of finite quantities. It is said, that the minutest errors are not to be neglected in mathematics: that the fluxions are celerities, not proportional to the finite increments though ever so small; but only to the moments of nascent increments, whereof the proportion alone, and not the magnitude, is considered. And of the aforesaid fluxions there be other fluxions, which fluxions of fluxions are called second fluxions. And the fluxions of these second fluxions are called third fluxions: and so on, fourth, fifth, sixth, &c., *ad infinitum*. Now as our sense is strained and puzzled with the perception of objects extremely minute, even so

* Introd. ad Quadraturam Curvarum.

the imagination, which faculty derives from sense, is very much strained and puzzled to frame clear ideas of the least particles of time, or the least increments generated therein: and much more so to comprehend the moments, or those increments of the flowing quantities in *statu nascenti*, in their very first origin or beginning to exist, before they become finite particles. And it seems still more difficult to conceive the abstracted velocities of such nascent, imperfect entities. But the velocities of the velocities, the second, third, fourth, and fifth velocities, &c., exceed, if I mistake not, all human understanding. The further the mind analyzeth and pursueth these fugitive ideas, the more it is lost and bewildered; the objects, at first fleeting and minute, soon vanishing out of sight. Certainly, in any sense, a second or third fluxion seems an obscure mystery. The incipient celerity of an incipient celerity, the nascent augment of a nascent augment, i. e. of a thing which hath no magnitude; take it in what light you please, the clear conception of it will, if I mistake not, be found impossible: whether it be so or no, I appeal to the trial of every thinking reader. And if a second fluxion be inconceivable, what are we to think of third, fourth, fifth fluxions, and so onward without end?

V. The foreign mathematicians are supposed by some, even of our own, to proceed in a manner less accurate perhaps and geometrical, yet more intelligible. Instead of flowing quantities and their fluxions, they consider the variable finite quantities, as increasing or diminishing by the continual addition or subduction of infinitely small quantities. Instead of the velocities wherewith increments are generated, they consider the increments or decrements themselves, which they call differences, and which are supposed to be infinitely small. The difference of a line is an infinitely little line; of a plane, an infinitely little plane. They suppose finite quantities to consist of parts infinitely little, and curves to be polygons, whereof the sides are infinitely little, which by the angles they make one with another determine the curvity of the line. Now to conceive a quantity infinitely small, that is, infinitely less than any sensible or imaginable quantity, or any the least finite magnitude, is, I confess, above my capacity. But to conceive a part of such infinitely small quantity, that shall be still infinitely less than it, and consequently though multiplied infinitely, shall never equal the minutest finite quantity, is, I suspect, an infinite difficulty to any man whatsoever; and will be allowed such by those who candidly say what they think; provided they really think and reflect, and do not take things upon trust.

VI. And yet in the *calculus differentialis*, which method serves to all the same intents and ends with that of fluxions, our modern analysts are not content to consider only the differences of finite quantities: they also consider the differences of those differences, and the differences of the differences of the first differences; and so on *ad infinitum.* That is, they consider quantities infinitely less than the least discernible quantity; and others infinitely less than those infinitely small ones; and still others infinitely less than the preceding infinitesimals, and so on without end or limit. Insomuch that we are to admit an infinite succession of infinitesimals, each infinitely less than the foregoing, and infinitely greater than the following. As there are first, second, third, fourth, fifth, &c., fluxions, so there are differences, first, second, third, fourth, &c., in an infinite progression towards nothing, which you still approach and never arrive at. And, which is most strange, although you should take a million of millions of these infinitesimals, each whereof is supposed infinitely greater than some other real magnitude, and add them to the least given quantity, it shall be never the bigger. For this is one of the modest *postulata* of our modern mathematicians, and is a cornerstone or groundwork of their speculations.

VII. All these points, I say, are supposed and believed by certain rigorous exactors of evidence in religion, men who pretend to believe no further than they can see. That men who have been conversant only about clear points should with difficulty admit obscure ones, might not seem altogether unaccountable. But he who can digest a second or third fluxion, a second or third difference, need not, methinks, be squeamish about any point in divinity. There is a natural presumption that men's faculties are made alike. It is on this supposition that they attempt to argue and convince one another. What, therefore, shall appear evidently impossible and repugnant to one may be presumed the same to another. But with what appearance of reason shall any man presume to say, that mysteries may not be objects of faith, at the same time that he himself admits such obscure mysteries to be the object of science?

VIII. It must indeed be acknowledged, the modern mathematicians do not consider these points as mysteries, but as clearly conceived and mastered by their comprehensive minds. They scruple not to say, that, by the help of these new analytics they can penetrate into infinity itself: that they can even extend their views beyond infinity: that their art comprehends not only infinite, but infinite of infinite (as they express it), or an infinity of infinites. But, notwithstanding all these assertions and pretensions, it may be justly questioned whether, as other men in other inquiries are often deceived by words or terms, so they likewise are not wonderfully deceived and deluded by their own peculiar signs, symbols, or species. Nothing is easier than to devise expressions or notations for fluxions and infinitesimals of the first, second, third, fourth, and subsequent orders, proceeding in the same regular form without end or limit, $\dot{x}$. $\ddot{x}$. $\dddot{x}$. $\ddddot{x}$. &c., or *dx. ddx. dddx. ddddx.*

&c. These expressions indeed are clear and distinct, and the mind finds no difficulty in conceiving them to be continued beyond any assignable bounds. But if we remove the veil and look underneath, if laying aside the expressions we set ourselves attentively to consider the things themselves, which are supposed to be expressed or marked thereby, we shall discover much emptiness, darkness, and confusion; nay, if I mistake not, direct impossibilities and contradictions. Whether this be the case or no, every thinking reader is entreated to examine and judge for himself.

IX. Having considered the object, I proceed to consider the principles of this new analysis by momentums, fluxions, or infinitesimals; wherein if it shall appear that your capital points, upon which the rest are supposed to depend, include error and false reasoning; it will then follow that you, who are at a loss to conduct yourselves, cannot with any decency set up for guides to other men. The main point in the method of fluxions is to obtain the fluxion or momentum of the rectangle or product of two indeterminate quantities. Inasmuch as from thence are derived rules for obtaining the fluxions of all other products and powers, be the co-efficients or the indexes what they will, integers or fractions, rational or surd. Now this fundamental point, one would think, should be very clearly made out, considering how much is built upon it, and that its influence extends throughout the whole analysis. But let the reader judge. This is given for demonstration.* Suppose the product or rectangle A B increased by continual motion: and that the momentaneous increments of the sides A and B are $a$ and $b$. When the sides A and B were deficient, or lesser by one-half of their moments, the rectangle was $\overline{A - \frac{1}{2}a} \times \overline{B - \frac{1}{2}b}$, i.e. $AB - \frac{1}{2}aB - \frac{1}{2}bA + \frac{1}{4}ab$. And as soon as the sides A and B are increased by the other two halves of their moments, the rectangle becomes $\overline{A+\frac{1}{2}a} \times \overline{B+\frac{1}{2}b}$ or $AB + \frac{1}{2}aB + \frac{1}{2}bA + \frac{1}{4}ab$. From the latter rectangle subduct the former, and the remaining difference will be $aB + bA$. Therefore the increment of the rectangle generated by the entire increments $a$ and $b$ is $aB + bA$. Q. E. D. But it is plain that the direct and true method to obtain the moment or increment of the rectangle AB, is to take the sides as increased by their whole increments, and so multiply them together, $A+a$ by $B+b$, the product whereof $AB + aB + bA + ab$ is the augmented rectangle; whence if we subduct AB, the remainder $aB + bA + ab$ will be the true increment of the rectangle, exceeding that which was obtained by the former illegitimate and indirect method by the quantity $ab$. And this holds universally by the quantities $a$ and $b$, be what they will, big or little, finite or infinitesimal, increments, moments, or velocities. Nor will it avail to say that $a\,b$

* Naturalis Philosophiæ Principia Mathematica, lib. ii. lem. 2.

is a quantity exceeding small: since we are told that *in rebus mathematicis errores quam minimi non sunt contemnendi.*

X. *Such reasoning as this for demonstration, nothing but the obscurity of the subject could have encouraged or induced the great author of the fluxionary method to put upon his followers, and nothing but an implicit deference to authority could move them to admit. The case indeed is difficult. There can be nothing done till you have got rid of the quantity *ab*. In order to this the notion of fluxions is shifted: it is placed in various lights: points which should be clear as first principles are puzzled; and terms which should be steadily used are ambiguous. But notwithstanding all this address and skill the point of getting rid of *a b* cannot be obtained by legitimate reasoning. If a man by methods not geometrical or demonstrative, shall have satisfied himself of the usefulness of certain rules; which he afterwards shall propose to his disciples for undoubted truths; which he undertakes to demonstrate in a subtile manner, and by the help of nice and intricate notions; it is not hard to conceive that such his disciples may, to save themselves the trouble of thinking, be inclined to confound the usefulness of a rule with the certainty of a truth, and accept the one for the other; especially if they are men accustomed rather to compute than to think; earnest rather to go on fast and far than solicitous to set out warily and see their way distinctly.

XI. The points or mere limits of nascent lines are undoubtedly equal, as having no more magnitude one than another, a limit, as such, being no quantity. If by a momentum you mean more than the initial limit, it must be either a finite quantity or an infinitesimal. But all finite quantities are expressly excluded from the notion of a momentum. Therefore the momentum must be an infinitesimal. And indeed, though much artifice hath been employed to escape or avoid the admission of quantities infinitely small, yet it seems ineffectual. For aught I see, you can admit no quantity as a medium between a finite quantity and nothing, without admitting infinitesimals. An increment generated in a finite particle of time is itself a finite particle; and cannot therefore be a momentum. You must therefore take an infinitesimal part of time wherein to generate your momentum. It is said, the magnitude of moments is not considered. And yet these same moments are supposed to be divided into parts. This is not easy to conceive, nor more than it is why we should take quantities less than A and B in order to obtain the increment of A B, of which proceeding it must be owned the final cause or motive is obvious; but it is not so obvious or easy to explain a just and legitimate reason for it, or show it to be geometrical.

XII. From the foregoing principle so demonstrated, the gene-

* Introd. ad Quadraturam Curvarum.

ral rule for finding the fluxion of any power of a flowing quantity is derived.* But as there seems to have been some inward scruple or consciousness of defect in the foregoing demonstration, and as this finding the fluxion of a given power is a point of primary importance, it hath therefore been judged proper to demonstrate the same in a different manner independent of the foregoing demonstration. But whether this other method be more legitimate and conclusive than the former, I proceed now to examine; and in order thereto shall premise the following lemma. "If with a view to demonstrate any proposition a certain point is supposed, by virtue of which certain other points are attained; and such supposed point be itself afterwards destroyed or rejected by a contrary supposition; in that case all the other points attained thereby and consequent thereupon, must also be destroyed and rejected, so as from thenceforward to be no more supposed or applied in the demonstration." This is so plain as to need no proof.

XIII. Now the other method of obtaining a rule to find the fluxion of any power is as follows. Let the quantity $x$ flow uniformly, and be it proposed to find the fluxion of $x^n$. In the same time that $x$ by flowing becomes $x + o$, the power $x^n$ becomes $\overline{x+o}|^n$, i. e. by the method of infinite series $x^n + nox^n - 1 + \frac{nn-n}{2} oox^{n-2} +$ &c., and the increments $o$ and $nox^n - 1 + \frac{nn-n}{2} oox^n - 2 +$ &c., are one to another as 1 to $nx^n - 1 + \frac{nn-n}{2} ox +$ &c. Let now the increments vanish, and their last proportion will be 1 to $nx^{n-1}$. But it should seem that this reasoning is not fair or conclusive. For when it is said, let the increments vanish, i. e. let the increments be nothing or let there be no increments, the former supposition that the increments were something, or that there were increments, is destroyed, and yet a consequence of that supposition, i. e. an expression got by virtue thereof, is retained. Which, by the foregoing lemma, is a false way of reasoning. Certainly when we suppose the increments to vanish, we must suppose their proportions, their expressions, and every thing else derived from the supposition of their existence, to vanish with them.

XIV. To make this point plainer, I shall unfold the reasoning, and propose it in a fuller light to your view. It amounts therefore to this, or may in other words be thus expressed. I suppose that the quantity $x$ flows, and by flowing is increased, and its increment I call $o$, so that by flowing it become $x + o$.

* Philosophiæ Naturalis Principia Mathematica, lib. ii. lem. 2.

And as $x$ increaseth, it follows that every power of $x$ is likewise increased in a due proportion. Therefore as $x$ becomes $x + o$, $x$ will become $\overline{x+o}\,|^n$: that is, according to the method of infinite series, $x^n + n\,o\,x^{n}-1 + \frac{nn-n}{2}o\,o\,x^{n-2} +$ &c. And if from the two augmented quantities we subduct the root and the power respectively, we shall have remaining the two increments, to wit, $o$ and $n\,o\,x^n - 1 + \frac{nn-n}{2}o\,o\,x^{n-2} +$ &c., which increments, being both divided by the common divisor $o$, yield the quotients 1 and $nx^n - 1 + \frac{nn-n}{2}o\,x^{n-2} +$ &c., which are therefore exponents of the ratio of the increments. Hitherto I have supposed that $x$ flows, that $x$ hath a real increment, that $o$ is something. And I have proceeded all along on that supposition, without which I should not have been able to have made so much as one single step. From that supposition it is that I get at the increment of $x^n$, that I am able to compare it with the increment of $x$, and that I find the proportion between the two increments. I now beg leave to make a new supposition contrary to the first, i. e. I will suppose that there is no increment of $x$, or that $o$ is nothing; which second supposition destroys my first, and is inconsistent with it, and therefore with every thing that supposeth it. I do nevertheless beg leave to retain $n\,x^n-1$, which is an expression obtained in virtue of my own supposition, which necessarily presupposed such supposition, and which could not be obtained without it. All which seems a most inconsistent way of arguing, and such as would not be allowed of in divinity.

XV. Nothing is plainer than that no just conclusion can be directly drawn from two inconsistent suppositions. You may indeed suppose any thing possible; but afterwards you may not suppose any thing that destroys what you first supposed. Or if you do, you must begin *de novo*. If, therefore, you suppose that the augments vanish, i. e. that there are no augments, you are to begin again, and see what follows from such supposition. But nothing will follow to your purpose. You cannot by that means ever arrive at your conclusion, or succeed in, what is called by the celebrated author, the investigation of the first or last proportions of nascent and evanescent quantities, by instituting the analysis in finite ones. I repeat it again: you are at liberty to make any possible supposition: and you may destroy one supposition by another: but then you may not retain the consequences, or any part of the consequences, of your first supposition so destroyed. I admit that signs may be made to denote either any thing or nothing: and, consequently, that in the original notation $x + o$, $o$ might have signified either an increment or nothing. But then which of these soever you make it signify, you must argue consistently with such its signification, and not proceed upon a double meaning: which to do were a manifest sophism. Whether you argue in symbols or in words, the rules of right reason are still the same. Nor can it be supposed, you will plead a privilege in mathematics to be exempt from them.

XVI. If you assume at first a quantity increased by nothing, and in the expression $x + o$, $o$ stands for nothing, upon this supposition as there is no increment of the root, so there will be no increment of the power; and, consequently, there will be none except the first, of all those members of the series constituting the power of the binomial; and will therefore never come to your expression of a fluxion legitimately by such method. Hence you are driven into the fallacious way of proceeding to a certain point on the supposition of an increment, and then at once shifting your supposition to that of no increment. There may seem great skill in doing this at a certain point or period. Since, if this second supposition had been made before the common division by $o$, all had vanished at once, and you must have got nothing by your supposition. Whereas, by this artifice of first dividing, and then changing your supposition, you retain 1 and $n x^n$ —1. But notwithstanding all this address to cover it, the fallacy is still the same. For whether it be done sooner or later, when once the second supposition or assumption is made, in the same instant the former assumption, and all that you got by it, is destroyed, and goes out together. And this is universally true, be the subject what it will, throughout all the branches of human knowledge; in any other of which, I believe men would hardly admit such a reasoning as this, which, in mathematics, is accepted for demonstration.

XVII. It may not be amiss to observe, that the method for finding the fluxion of a rectangle of two flowing quantities, as it is set forth in the Treatise of Quadratures, differs from the above-mentioned taken from the second book of the Principles, and is, in effect, the same with that used in the *Calculus Differentialis.** For the supposing a quantity infinitely diminished, and therefore rejecting it, is, in effect, the rejecting an infinitesimal; and, indeed, it requires a marvellous sharpness of discernment, to be able to distinguish between evanescent increments and infinitesimal differences. It may, perhaps, be said, that the quantity being infinitely diminished becomes nothing, and so nothing is rejected. But according to the received principles it is evident, that no geometrical quantity can, by any division or subdivision whatsoever, be exhausted or reduced to nothing.

* Analyse des Infiniments Petits, part i. prop. 2.

Considering the various arts and devices used by the great author of the fluxionary method, in how many lights he placeth his fluxions, and in what different ways he attempts to demonstrate the same point; one would be inclined to think, he was himself suspicious of the justness of his own demonstrations, and that he was not enough pleased with any one notion steadily to adhere to it. Thus much at least is plain, that he owned himself satisfied concerning certain points, which nevertheless he would not undertake to demonstrate to others.* Whether this satisfaction arose from tentative methods or inductions, which have often been admitted by mathematicians (for instance by Dr. Wallis, in his Arithmetic of Infinites), is what I shall not pretend to determine. But whatever the case might have been with respect to the author, it appears that his followers have shown themselves more eager in applying his method, than accurate in examining his principles.

XVIII. It is curious to observe, what subtlety and skill this great genius employs to struggle with an insuperable difficulty; and through what labyrinths he endeavours to escape the doctrine of infinitesimals; which, as it intrudes upon him whether he will or no, so it is admitted and embraced by others without the least repugnance; Leibnitz and his followers, in their *calculus differentialis*, making no manner of scruple, first to suppose, and secondly to reject, quantities infinitely small: with what clearness in the apprehension, and justness in the reasoning, any thinking man, who is not prejudiced in favour of those things, may easily discern. The notion or idea of an infinitesimal quantity, as it is an object simply apprehended by the mind, hath been already considered.† I shall now only observe, as to the method of getting rid of such quantities, that it is done without the least ceremony. As in fluxions, the point of first importance, and which paves the way to the rest, is to find the fluxion of a product of two indeterminate quantities, so in the *calculus differentialis* (which method is supposed to have been borrowed from the former with some small alterations), the main point is to obtain the difference of such product. Now the rule for this is got by rejecting the product or rectangle of the differences. And in general it is supposed, that no quantity is bigger or lesser for the addition or subduction of its infinitesimal; and, consequently, no error can arise from such rejection of infinitesimals.

XIX. And yet it should seem that, whatever errors are admitted in the premises, proportional errors ought to be apprehended in the conclusion, be they finite or infinitesimal: and therefore the ἀκριβεία of geometry requires nothing should be

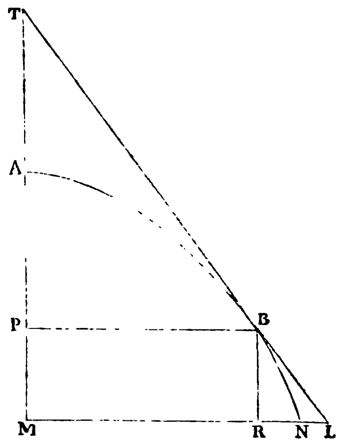
* See Letter to Collins, Nov 9, 1679.

† Sect. v. and vi.

neglected or rejected. In answer to this you will perhaps say, that the conclusions are accurately true, and that therefore the principles and methods from whence they are derived must be so too. But this inverted way of demonstrating your principles by your conclusions, as it would be peculiar to you gentlemen, so it is contrary to the rules of logic. The truth of the conclusion will not prove either the form or the matter of a syllogism to be true; inasmuch as the illation might have been wrong or the premises false, and the conclusion nevertheless true, though not in virtue of such illation or of such premises. I say that in every other science men prove their conclusions by their principles, and not their principles by the conclusions. But if in yours you should allow yourselves this unnatural way of proceeding, the consequence would be that you must take up with induction, and bid adieu to demonstration. And if you submit to this, your authority will no longer lead the way in points of reason and science.

XX. I have no controversy about your conclusions, but only about your logic and method: how you demonstrate; what objects you are conversant with, and whether you conceive them clearly; what principles you proceed upon; how sound they may be; and how you apply them. It must be remembered that I am not concerned about the truth of your theorems, but only about the way of coming at them; whether it be legitimate or illegitimate, clear or obscure, scientific or tentative. To prevent all possibility of your mistaking me, I beg leave to repeat and insist, that I consider the geometrical analyst as a logician, i. e. so far forth as he reasons and argues, and his mathematical conclusions, not in themselves, but in their premises; not as true or false, useful or insignificant, but as derived from such principles, and by such inferences. And forasmuch as it may perhaps seem an unaccountable paradox, that mathematicians should deduce true propositions from false principles, be right in the conclusion, and yet err in the premises; I shall endeavour particularly to explain why this may come to pass, and show how error may bring forth truth, though it cannot bring forth science.

XXI. In order therefore to clear up this point, we will suppose, for instance, that a tangent is to be drawn to a parabola, and examine the progress of this affair, as it is performed by infinitesimal differences. Let A B be a curve, the absciss A P $= x$, the ordinate P B $= y$, the difference of the absciss P M $= dx$, the difference of the ordinate R N $= dy$. Now by supposing the curve to be a polygon, and consequently B N, the increment or difference of the curve, to be a straight line coincident with the tangent, and the differential triangle B R N to be similiar to the triangle T P B, the subtangent P T is found a fourth proportional to R N : R B : P B: that is to $dy : dx : y$. Hence the subtangent will be $\frac{y\,dx}{dy}$. But

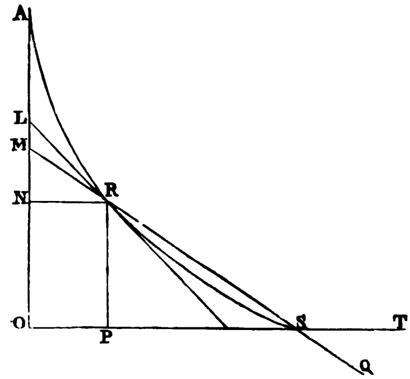



herein there is an error arising from the forementioned false supposition, whence the value of P T comes out greater than the truth: for in reality it is not the triangle R N B, but R L B, which is similar to P B T, and therefore (instead of R N) R L should have been the first term of the proportion, i. e. R N + N L, i. e. $dy + z$: whence the true expression for the subtangent should have been $\frac{y\,dx}{dy+z}$. There was therefore an error of defect in making $dy$ the divisor: which error was equal to $z$, i. e. N L the line comprehended between the curve and the tangent. Now by the nature of the curve $yy = px$, supposing $p$ to be the parameter, whence by the rule of differences $2y\,dy = p\,dx$ and $dy = \frac{p\,dx}{2y}$. But if you multiply $y + dy$ by itself, and retain the whole product without rejecting the square of the difference, it will then come out, by substituting the augmented quantities in the equation of the curve, that $dy = \frac{p\,dx}{2y} - \frac{dy\,dy}{2y}$ truly. There was therefore an error of excess in making $dy = \frac{p\,dx}{2y}$, which followed from the erroneous rule of differences. And the measure of this second error is $\frac{dy\,dy}{2y} = z$. Therefore the two errors, being equal and contrary, destroy each other; the first error of defect being corrected by a second error of excess.

XXII. If you had committed only one error, you would not have come at a true solution of the problem. But by virtue of a twofold mistake you arrive, though not at science, yet at truth. For science it cannot be called, when you proceed blindfold, and arrive at the truth not knowing how or by what means. To demonstrate that $z$ is equal to $\frac{dy\,dy}{2y}$, let B R or $dx$ be $m$, and R N or $dy$ be $n$. By the thirty-third proposition of the first book of the Conics of Apollonius, and from similar triangles, as $2x$ to $y$ so is $m$ to $n + z = \frac{my}{2x}$. Likewise from the nature of the parabola $yy + 2yn + nn = xp + mp$, and $2yn + nn = mp$: wherefore $\frac{2yn + nn}{p} = m$: and because $yy = px$, $\frac{yy}{p}$ will be equal to $x$. Therefore substituting these values instead of $m$ and $x$ we shall have $n + z = \frac{my}{2x} = \frac{2yynp + ynnp}{2yyp}$: i. e. $n + z = \frac{2yn + nn}{2y}$: which being reduced gives $z = \frac{nn}{2y} = \frac{dy\,dy}{2y}$ Q. E. D.

XXIII. Now I observe in the first place, that the conclusion comes out right, not because the rejected square of $dy$ was infinitely small; but because this error was compensated by another contrary and equal error. I observe in the second place, that whatever is rejected, be it ever so small, if it be real, and consequently makes a real error in the premises, it will produce a proportional real error in the conclusion. Your theorems therefore cannot be accurately true, nor your problems accurately solved, in virtue of premises which themselves are not accurate: it being a rule in logic that *conclusio sequitur partem debiliorem.* Therefore I observe in the third place, that when the conclusion is evident and the premises obscure, or the conclusion accurate and the premises inaccurate, we may safely pronounce that such conclusion is neither evident nor accurate, in virtue of those obscure, inaccurate premises or principles; but in virtue of some other principles which perhaps the demonstrator himself never knew or thought of. I observe in the last place, that in case the differences are supposed finite quantities ever so great, the conclusion will nevertheless come out the same; inasmuch as the rejected quantities are legitimately thrown out, not for their smallness, but for another reason, to wit, because of contrary errors, which destroying each other do upon the whole cause that nothing is really, though something is apparently, thrown out. And this reason holds equally with respect to quantities finite as well as infinitesimal, great as well as small, a foot or a yard long, as well as the minutest increment.

XXIV. For the fuller illustration of this point, I shall consider it in another light, and proceeding in finite quantities to the conclusion, I shall only then make use of one infinitesimal. Suppose the straight line M Q cuts the curve A T in the points R and S. Suppose L R a tangent at the point R, A N the



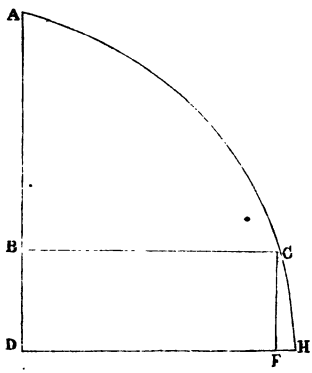

abscis, N R and O S ordinates. Let A N be produced to O, and R P be drawn parallel to N O. Suppose A N = $x$, N R = $y$, N O = $v$, P S = $z$, the subsecant M N = S. Let the equation $y = x\,x$ express the nature of the curve: and supposing $y$ and $x$ increased by their finite increments, we get $y + z = xx + 2\,x\,v + v\,v$: whence the former equation being subducted, there remains $z = 2\,x\,v + v\,v$. And by reason of similar triangles P S : P R :: N R : N M, i. e. $z : v :: y : s = \frac{v\,y}{z}$, wherein if for $y$ and $z$ we substitute their values, we get $\frac{vxx}{2x\,v + v\,v} = s = \frac{xx}{2x + v}$. And supposing N O to be infinitely diminished, the subsecant N M will in that case coincide with the subtangent N L, and $v$ as an infinitesimal may be rejected: whence it follows that S = N L = $\frac{xx}{2x} = \frac{x}{2}$; which is the true value of the subtangent. And since this was obtained by one only error, i. e. by once rejecting one only infinitesimal, it should seem, contrary to what hath been said, that an infinitesimal quantity or difference may be neglected or thrown away, and the conclusion nevertheless be accurately true, although there was no double mistake or rectifying of one error by another, as in the first case. But if this point be thoroughly considered, we shall find there is even here a double mistake, and that one compensates or rectifies the other. For in the first place, it was supposed, that when N O is infinitely diminished or becomes an infinitesimal, then the subsecant N M becomes equal to the subtangent N L. But this is a plain mistake; for it is evident, that as a secant cannot be a tangent, so a subsecant cannot be a subtangent. Be the difference ever so small, yet still there is a difference. And if N O be infinitely small, there will even then be an infinitely small difference between N M and N L. Therefore N M or S was too little for your supposition (when you supposed it equal to N L), and this error was compensated by a second error in throwing out $v$, which last error made $s$ bigger than its true value, and in lieu thereof gave the value of the subtangent. This is the true state of the case, however it may be disguised. And to this in reality it amounts, and is at bottom the same thing, if we should pretend to find the subtangent by having first found, from the equation of the curve and similar triangles, a general expression for all subsecants, and then reducing the subtangent under this general rule, by considering it as the subsecant when $v$ vanishes, or becomes nothing.

XXV. Upon the whole I observe, first, that $v$ can never be nothing, so long as there is a secant. Secondly, that the same line cannot be both tangent and secant. Thirdly, that when $v$ or N O* vanisheth, P S and S R do also vanish, and with them the proportionality of the similar triangles. Consequently the whole expression, which was obtained by means thereof and grounded thereupon, vanisheth when $v$ vanisheth. Fourthly, that the method for finding secants or the expression of secants, be it ever so general, cannot in common sense extend any further than to all secants whatsoever; and, as it necessarily supposed similar triangles, it cannot be supposed to take place where there are not similar triangles. Fifthly, that the subsecant will always be less than the subtangent, and can never coincide with it; which coincidence to suppose would be absurd: for it would be supposing the same line at the same time to cut and not to cut another given line, which is a manifest contradiction, such as subverts the hypothesis and gives a demonstration of its falsehood. Sixthly, if this be not admitted, I demand a reason why any other apagogical demonstration, or demonstration *ad absurdum*, should be admitted in geometry rather than this; or that some real difference be assigned between this and others as such. Seventhly, I observe that it is sophistical to suppose N O or R P, P S, and S R to be finite real lines in order to form the triangle R P S, in order to obtain proportions by similar triangles; and afterwards to suppose there are no such lines, nor consequently similar triangles, and nevertheless to retain the consequence of the first supposition, after such supposition hath been destroyed by a contrary one. Eighthly, that although in the present case, by inconsistent suppositions truth may be obtained, yet such truth is not demonstrated; that such method is not conformable to the rules of logic and right reason; that, however useful it may be, it must be considered only as a presumption, as a knack, an art, rather an artifice, but not a scientific demonstration.

XXVI. The doctrine premised may be further illustrated by the following simple and easy case, wherein I shall proceed by evanescent increments. Suppose A B = $x$, B C = $y$, B D = $o$,

* See the foregoing figure.

and that $xx$ is equal to the area A B C: it is proposed to find the ordinate $y$ or B C. When $x$ by flowing becomes $x + o$, then




$x x$ becomes $x x + 2\ x\ o + o\ o$: and the area A B C becomes A D H, and the increment of $xx$ will be equal to B D H C the increment of the area, i. e. to B C F D + C F H. And if we suppose the curvilinear space C F H to be $q\ o\ o$, then $2\ x\ o + o\ o = y\ o + q\ o\ o$, which divided by $o$ gives $2\ x + o = y + q\ o$. And, supposing $o$ to vanish, $2\ x = y$, in which case A C H will be a straight line, and the areas A B C, C F H, triangles. Now with regard to this reasoning, it hath been already remarked,* that it is not legitimate or logical to suppose $o$ to vanish, i. e. to be nothing, i. e. that there is no increment, unless we reject at the same time with the increment itself every consequence of such increment, i. e. whatsoever could not be obtained but by supposing such increment. It must nevertheless be acknowledged, that the problem is rightly solved, and the conclusion true, to which we are led by this method. It will therefore be asked, how comes it to pass that the throwing out $o$ is attended with no error in the conclusion? I answer, the true reason hereof is plainly this: because $q$ being unit, $q\ o$ is equal to $o$: and therefore $2\ x + o - qo = y = 2\ x$, the equal quantities $q\ o$ and $o$ being destroyed by contrary signs.

XXVII. As on the one hand it were absurd to get rid of $o$ by saying, let me contradict myself; let me subvert my own hypothesis; let me take it for granted that there is no increment, at the same time that I retain a quantity, which I could never have got at but by assuming an increment: so on the other hand it would be equally wrong to imagine, that in a geometrical demonstration we may be allowed to admit any error, though ever so small, or that it is possible, in the nature of things, an accurate conclusion should be derived from inaccurate principles.

* Sect xiii. and xiii. supra.

Therefore $o$ cannot be thrown out as an infinitesimal, or upon the principle that infinitesimals may be safely neglected; but only because it is destroyed by an equal quantity with a negative sign, whence $o - po$ is equal to nothing. And as it is illegitimate to reduce an equation, by subducting from one side a quantity when it is not to be destroyed, or when an equal quantity is not subducted from the other side of the equation: so it must be allowed a very logical and just method of arguing, to conclude that if from equals either nothing or equal quantities are subducted, they shall still remain equal. And this is a true reason why no error is at last produced by the rejecting of $o$. Which therefore must not be ascribed to the doctrine of differences, or infinitesimals, or evanescent quantities, or momentums, or fluxions.

XXVIII. Suppose the case to be general, and that $x^n$ is equal to the area A B C, whence by the method of fluxions the ordinate is found $nx^{n-1}$, which we admit for true, and shall inquire how it is arrived at. Now if we are content to come at the conclusion in a summary way, by supposing that the ratio of the fluxions of $x$ and $x^n$ is found* to be 1 and $n\,x^{n-1}$, and that the ordinate of the area is considered as its fluxion; we shall not so clearly see our way, or perceive how the truth comes out, that method, as we have shown before, being obscure and illogical. But if we fairly delineate the area and its increment, and divide the latter in two parts B C F D and C F H,† and proceed regularly by equations between the algebraical and geometrical quantities, the reason of the thing will plainly appear. For as $x^n$ is equal to the area A B C, so is the increment of $x^n$ equal to the increment of the area, i. e. to B D H C; that is to say $n\,o\,x^{n-1} + \frac{nn-n}{2}o\,o\,x^{n-2} + \&c. = BDFC + CFH$. And only the first members on each side of the equation being retained, $n\,o\,x^{n-1} = B\,D\,F\,C$: and dividing both sides by $o$ or B D, we shall get $n\,x^{n-1} = B\,C$. Admitting therefore; that the curvilinear space C F H is equal to the rejectaneous quantity $\frac{nn-n}{2}o\,o\,x^{n-2} + \&c.$; and that when this is rejected on one side, that is rejected on the other, the reasoning becomes just and the conclusion true. And it is all one whatever magnitude you allow to B D, whether that of an infinitesimal difference or a finite increment ever so great. It is therefore plain, that the supposing the rejectaneous algebraical quantity to be an infinitely small or evanescent quantity, and therefore to be neglected, must have produced an error, had it not been for the curvilinear spaces being equal thereto, and at the same time subducted from the other part or side of the equation, agreeably to the axiom: 'If

* Sect. xiii.

† See the figure in Sect. xxvi.

from equals you subduct equals, the remainders will be equal. For those quantities, which by the analysts are said to be neglected, or made to vanish, are in reality subducted. If therefore the conclusion be true, it is absolutely necessary that the finite space C F H be equal to the remainder of the increment expressed by $\frac{nn-n}{2}\, o\, o\, x^{n-2}$ &c., equal, I say, to the finite remainder of a finite increment.

XXIX. Therefore, be the power what you please, there will arise on one side an algebraical expression, on the other a geometrical quantity, each of which naturally divides itself into three members; the algebraical or fluxionary expression into one, which includes neither the expression of the increment of the absciss nor of any power thereof; another which includes the expression of the increment itself; and a third including the expression of the powers of the increment. The geometrical quantity also or whole increased area consists of three parts or members, the first of which is the given areas, the second a rectangle under the ordinate and the increment of the absciss, and the third a curvilinear space. And, comparing the homologous or correspondent members on both sides, we find that as the first member of the expression is the expression of the given area, so the second member of the expression will express the rectangle or second member of the geometrical quantity; and the third, containing the powers of the increment, will express the curvilinear space or third member of the geometrical quantity. This hint may perhaps be further extended, and applied to good purpose, by those who have leisure and curiosity for such matters. The use I make of it is to show, that the analysis cannot obtain in augments or differences, but it must also obtain in finite quantities, be they ever so great, as was before observed.

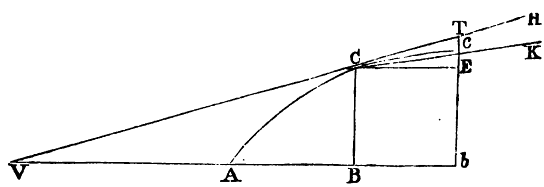
XXX. It seems therefore upon the whole, that we may safely pronounce the conclusion cannot be right, if in order thereto any quantity be made to vanish, or be neglected, except that either one error is redressed by another; or that, secondly, on the same side of an equation equal quantities are destroyed by contrary signs, so that the quantity we mean to reject is first annihilated; or lastly, that from the opposite sides equal quantities are subducted. And therefore to get rid of quantities by the received principles of fluxions or of differences is neither good geometry nor good logic. When the augments vanish, the velocities also vanish. The velocities or fluxions are said to be *primò* and *ultimò*, as the augments nascent and evanescent. Take therefore the ratio of the evanescent quantities, it is the same with that of the fluxions: it will therefore answer all intents as well. Why then are fluxions introduced? Is it not to shun or rather to palliate the use of quantities infinitely small? But we have no notion whereby to conceive and measure various degrees of velocity, besides space and time, or when the times are given, besides space alone. We have even no notion of velocity prescinded from time and space. When therefore a point is supposed to move in given times, we have no notion of greater or lesser velocities or of proportions between velocities, but only of longer or shorter lines, and of proportions between such lines generated in equal parts of time.

XXXI. A point may be the limit of a line: a line may be the limit of a surface: a moment may terminate time. But how can we conceive a velocity by the help of such limits? It necessarily implies both time and space, and cannot be conceived without them. And if the velocities of nascent and evanescent quantities, i. e. abstracted from time and space, may not be comprehended, how can we comprehend and demonstrate their proportions; or consider their *rationes primæ* and *ultimæ?* For to consider the proportion or *ratio* of things implies that such things have magnitude; that such their magnitudes may be measured, and their relations to each other known. But as there is no measure of velocity except time and space, the proportion of velocities being only compounded of the direct proportion of the spaces and the reciprocal proportion of the times; doth it not follow that to talk of investigating, obtaining, and considering the proportions of velocities, exclusively of time and space, is to talk unintelligibly?

XXXII. But you will say that, in the use and application of fluxions, men do not overstrain their faculties to a precise conception of the above-mentioned velocities, increments, infinitesimals, or any other such like ideas of a nature so nice, subtile, and evanescent. And therefore you will perhaps maintain, that problems may be solved without those inconceivable suppositions; and that, consequently, the doctrine of fluxions, as to the practical part, stands clear of all such difficulties. I answer, that if in the use or application of this method those difficult and obscure points are not attended to, they are nevertheless supposed. They are the foundations on which the moderns build the principles on which they proceed, in solving problems and discovering theorems. It is with the method of fluxions as with all other methods, which presuppose their respective principles and are grounded thereon; although the rules may be practised by men who neither attend to, nor perhaps know, the principles. In like manner, therefore, as a sailor may practically apply certain rules derived from astronomy and geometry, the principles whereof he doth not understand: and as any ordinary man may solve divers numerical questions by the vulgar rules and operations of arithmetic, which he performs and applies without knowing the reasons of them: even so it cannot be denied that you may apply the rules of the fluxionary method: you may compare and reduce particular cases to the general forms; you may operate, and compute, and solve problems thereby, not only without an actual attention to, or an actual knowledge of, the grounds of that method, and the principles whereon it depends, and whence it is deduced, but even without having ever considered or comprehended them.

XXXIII. But then it must be remembered, that in such case, although you may pass for an artist, computist, or analyst, yet you may not be justly esteemed a man of science and demonstration. Nor should any man, in virtue of being conversant in such obscure analytics, imagine his rational faculties to be more improved than those of other men, which have been exercised in a different manner, and on different subjects; much less erect himself into a judge and an oracle, concerning matters that have no sort of connexion with, or dependence on, those species, symbols, or signs, in the management whereof he is so conversant and expert. As you, who are a skilful computist or analyst, may not therefore be deemed skilful in anatomy; or *vice versâ*, as a man who can dissect with art, may, nevertheless, be ignorant in your art of computing: even so you may both, notwithstanding your peculiar skill in your respective arts, be alike unqualified to decide upon logic, or metaphysics, or ethics, or religion. And this would be true, even admitting that you understood your own principles and could demonstrate them.

XXXIV. If it is said, that fluxions may be expounded or expressed by finite lines proportional to them; which finite lines, as they may be distinctly conceived, and known, and reasoned upon, so they may be substituted for the fluxions, and their mutual relations or proportions be considered as the proportions of fluxions; by which means the doctrine becomes clear and useful: I answer that if, in order to arrive at these finite lines proportional to the fluxions, there be certain steps made use of which are obscure and inconceivable, be those finite lines themselves ever so clearly conceived, it must nevertheless be acknowledged, that your proceeding is not clear, nor your method scientific. For instance, it is supposed that A B being



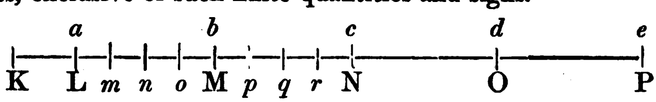

the absciss, B C the ordinate, and V C H a tangent of the curve A C, B *b* or C E the increment of the absciss, E *c* the increment of the ordinate, which produced meets V H in the point T, and C *c* the increment of the curve. The right line C *c* being produced to K, there are formed three small triangles, the rectilinear C E *c*, the mixtilinear C E *c*, and the rectilinear triangle C E T. It is evident these three triangles are different from each other, the rectilinear C E *c* being less than the mixtilinear C E *c*, whose sides are the three increments above mentioned, and this still less than the triangle C E T. It is supposed that the ordinate *b c* moves into the place B C, so that the point *c* is coincident with the point C; and the right line C K, and consequently the curve C *c*, is coincident with the tangent C H. In which case the mixtilinear evanescent triangle C E *c* will, in its last form, be similar to the triangle C E T: and its evanescent sides C E, E *c*, and C *c* will be proportional to C E, E T, and C T, the sides of the triangle C E T. And therefore it is concluded, that the fluxions of the lines A B, B C, and A C, being in the last ratio of their evanescent increments, are proportional to the sides of the triangle C E T, or, which is all one, of the triangle V B C, similar thereunto. * It is particularly remarked and insisted on by the great author, that the points C and *c* must not be distant one from another, by any the least interval whatsoever: but that, in order to find the ultimate proportions of the lines C E, E *c*, and C *c* (i. e. the proportions of the fluxions or velocities), expressed by the finite sides of the triangle V B C, the points C and *c* must be accurately coincident, i. e., one and the same. A point therefore is considered as a triangle, or a triangle is supposed to be formed in a point. Which to conceive seems quite impossible. Yet some there are, who, though they shrink at all other mysteries, make no difficulty of their own, who "strain at a gnat and swallow a camel."

XXXV. I know not whether it be worth while to observe, that possibly some men may hope to operate by symbols and suppositions, in such sort as to avoid the use of fluxions, momentums, and infinitesimals, after the following manner. Suppose $x$ to be one absciss of a curve, and $z$ another absciss of the same curve. Suppose also that the respective areas are $xxx$ and $zzz$: and that $z - x$ is the increment of the absciss, and $zzz - xxx$ the increment of the area, without considering how great or how small those increments may be. Divide now $zzz - xxx$ by $z - x$ and the quotient will be $zz + zx + xx$: and, supposing that $z$ and $x$ are equal, this same quotient will be $3xx$, which in that case is the ordinate, which therefore may be thus obtained independently of fluxions and infinitesimals. But herein is a direct fallacy: for, in the first place, it is supposed that the abscisses $z$ and $x$ are unequal, without which supposition no one step could have been made; and, in the second

* Introd. ad Quadraturam Curvarum.

place, it is supposed they are equal; which is a manifest inconsistency, and amounts to the same thing that hath been before considered.* And there is indeed reason to apprehend, that all attempts for setting the abstruse and fine geometry on a right foundation, and avoiding the doctrine of velocities, momentums, &c., will be found impracticable, till such time as the object and end of geometry are better understood, than hitherto they seem to have been. The great author of the method of fluxions felt this difficulty, and therefore he gave in to those nice abstractions and geometrical metaphysics, without which he saw nothing could be done on the received principles; and what in the way of demonstration he hath done with them the reader will judge. It must, indeed, be acknowledged, that he used fluxions, like the scaffold of a building, as things to be laid aside or got rid of, as soon as finite lines were found proportional to them. But then these finite exponents are found by the help of fluxions. Whatever, therefore, is got by such exponents and proportions is to be ascribed to fluxions: which must therefore be previously understood. And what are these fluxions? The velocities of evanescent increments? And what are these same evanescent increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?

XXXVI. Men too often impose on themselves and others, as if they conceived and understood things expressed by signs, when in truth they have no idea, save only of the very signs themselves. And there are some grounds to apprehend that this may be the present case. The velocities of evanescent or nascent quantities are supposed to be expressed, both by finite lines of a determinate magnitude, and by algebraical notes or signs: but I suspect that many who, perhaps never having examined the matter, take it for granted, would upon a narrow scrutiny find it impossible to frame any idea or notion whatsoever of those velocities, exclusive of such finite quantities and signs.




Suppose the line K P described by the motion of a point continually accelerated, and that in equal particles of time the unequal parts K L, L M, M N, N O, &c., are generated. Suppose also that *a*, *b*, *c*, *d*, *e*, &c., denote the velocities of the generating point, at the several periods of the parts or increments so generated. It is easy to observe, that these increments are each proportional to the sum of the velocities with which it is described: that, consequently, the several sums of the velocities,

* Sect. xv.

generated in equal parts of time, may be set forth by the respective lines K L, L M, M N, &c., generated in the same times: it is likewise an easy matter to say, that the last velocity generated in the first particle of time, may be expressed by the symbol $a$, the last in the second by $b$, the last generated in the third by $c$, and so on: that $a$ is the velocity of L M in *statu nascenti*, and $b$, $c$, $d$, $e$, &c., are the velocities of the increments M N, N O, O P, &c., in their respective nascent estates. You may proceed, and consider these velocities themselves as flowing or increasing quantities, taking the velocities of the velocities, and the velocities of the velocities of the velocities, i. e., the first, second, third, &c., velocities, *ad infinitum:* which succeeding series of velocities may be thus expressed, $a$. $b - a$. $c - 2\,b + a$. $d - 3\,c + 3\,b - a$, &c., which you may call by the names of first, second, third, fourth fluxions. And for an apter expression you may denote the variable flowing line K L, K M, K N, &c., by the letter $x$; and the first fluxions by $\dot{x}$, the second by $\ddot{x}$, the third by $\dddot{x}$, and so on, *ad infinitum.*

XXXVII. Nothing is easier than to assign names, signs, or expressions to these fluxions, and it is not difficult to compute and operate by means of such signs. But it will be found much more difficult to omit the signs, and yet retain in our minds the things which we suppose to be signified by them. To consider the exponents, whether geometrical, or algebraical, or fluxionary, is no difficult matter. But to form a precise idea of a third velocity, for instance, in itself and by itself, *hoc opus, hic labor.* Nor, indeed, is it an easy point to form a clear and distinct idea of any velocity at all, exclusive of and prescinding from all length of time and space; as also from all notes, signs, or symbols whatsoever. This, if I may be allowed to judge of others by myself, is impossible. To me it seems evident, that measures and signs are absolutely necessary, in order to conceive or reason about velocities; and that, consequently, when we think to conceive the velocities, simply and in themselves, we are deluded by vain abstractions.

XXXVIII. It may perhaps be thought by some an easier method of conceiving fluxions, to suppose them the velocities wherewith the infinitesimal differences are generated. So that the first fluxions shall be the velocities of the first differences, the second the velocities of the second differences, the third fluxions the velocities of the third differences, and so on, *ad infinitum.* But, not to mention the insurmountable difficulty of admitting or conceiving infinitesimals, and infinitesimals of infinitesimals, &c., it is evident that this notion of fluxions would not consist with the great author's view; who held that the minutest quantity ought not to be neglected, that therefore the doctrine of infinitesimal differences was not to be admitted in geometry; and who plainly appears to have introduced the use of velocities or fluxions, on purpose to exclude or do without them.

XXXIX. To others it may possibly seem, that we should form a juster idea of fluxions, by assuming the finite, unequal, isochronal increments K L, L M, M N, &c., and considering them in *statu nascenti*, also their increments in *statu nascenti*, and the nascent increments of those increments, and so on, supposing the first nascent increments to be proportional to the first fluxions or velocities, the nascent increments of those increments to be proportional to the second fluxions, the third nascent increments to be proportional to the third fluxions, and so onwards. And, as the first fluxions are the velocities of the first nascent increments, so the second fluxions may be conceived to be the velocities of the second nascent increments, rather than the velocities of velocities. By which means the analogy of fluxions may seem better preserved, and the notion rendered more intelligible.

XL. And indeed it should seem, that in the way of obtaining the second or third fluxion of an equation, the given fluxions were considered rather as increments than velocities. But the considering them sometimes in one sense, sometimes in another, one while in themselves, another in their exponents, seems to have occasioned no small share of that confusion and obscurity which is found in the doctrine of fluxions. It may seem, therefore, that the notion might be still mended, and that instead of fluxions of fluxions, or fluxions of fluxions of fluxions, and instead of second, third, or fourth, &c., fluxions of a given quantity, it might be more consistent and less liable to exception to say, the fluxion of the first nascent increment, i. e., the second fluxion; the fluxion of the second nascent increment, i. e., the third fluxion; the fluxion of the third nascent increment, i. e., the fourth fluxion, which fluxions are conceived respectively proportional, each to the nascent principle of the increment succeeding that whereof it is the fluxion.

XLI. For the more distinct conception of all which it may be considered, that if the finite increment L M* be divided into the isochronal parts L *m*, *m n*, *n o*, *o* M; and the increment M N into the parts M *p*, *p q*, *q r*, *r* N, isochronal to the former; as the whole increments L M, M N, are proportional to the sums of their describing velocities, even so the homologous particles L *m*, M *p* are also proportional to the respective accelerated velocities with which they are described. And as the velocity with which M *p* is generated, exceeds that with which L *m* was generated, even so the particle M *p* exceeds the particle L *m*. And in general, as the isochronal velocities describ-

* See the foregoing scheme in sect. xxxvi.

ing the particles of M N exceed the isochronal velocities describing the particles of L M, even so the particles of the former exceed the correspondent particles of the latter. And this will hold, be the said particles ever so small. M N therefore will exceed L M if they are both taken in their nascent states: and that excess will be proportional to the excess of the velocity *b* above the velocity *a*. Hence we may see that this last account of fluxions comes, in the upshot, to the same thing with the first.*

XLII. But notwithstanding what hath been said, it must still be acknowledged, that the finite particles L *m* or M *p*, though taken ever so small, are not proportional to the velocities *a* and *b;* but each to a series of velocities changing every moment, or which is the same thing, to an accelerated velocity, by which it is generated, during a certain minute particle of time: that the nascent beginnings or evanescent endings of finite quantities, which are produced in moments or infinitely small parts of time, are alone proportional to given velocities: that therefore, in order to conceive the first fluxions, we must conceive time divided into moments, increments generated in those moments, and velocities proportional to those increments: that in order to conceive second and third fluxions, we must suppose that the nascent principles or momentaneous increments have themselves also other momentaneous increments, which are proportional to their respective generating velocities; that the velocities of these second momentaneous increments are second fluxions: those of their nascent momentaneous increments third fluxions. And so on *ad infinitum.*

XLIII. By subducting the increment generated in the first moment from that generated in the second, we get the increment of an increment. And by subducting the velocity generating in the first moment from that generating in the second, we get the fluxion of a fluxion. In like manner, by subducting the difference of the velocities generating in the two first moments, from the excess of the velocity in the third above that in the second moment, we obtain the third fluxion. And after the same analogy we may proceed to fourth, fifth, sixth fluxions, &c. And if we call the velocities of the first, second, third, fourth moments, *a, b, c, d,* the series of fluxions will be as above, $a.\ b - a.\ c - 2b + a.\ d - 3c + 3b - a.$ *ad infinitum,* i. e. $\dot{x}.\ \ddot{x}.\ \dddot{x}.\ \ddddot{x}.$ *ad infinitum.*

XLIV. Thus fluxions may be considered in sundry lights and shapes, which seem all equally difficult to conceive. And indeed, as it is impossible to conceive velocity without time or space, without either finite length or finite duration,† it must seem above the powers of men to comprehend even the first fluxions. And if the first are incomprehensible, what shall we say of the second

* Sect. xxxvi. † Sect. xxxi.

and third fluxions, &c.? He who can conceive the beginning of a beginning, or the end of an end, somewhat before the first or after the last, may be perhaps sharp-sighted enough to conceive these things. But most men will, I believe, find it impossible to understand them in any sense whatever.

XLV. One would think that men could not speak too exactly on so nice a subject. And yet, as was before hinted, we may often observe that the exponents of fluxions, or notes representing fluxions, are confounded with the fluxions themselves. Is not this the case, when just after the fluxions of flowing quantities were said to be the celerities of their increasing, and the second fluxions to be the mutations of the first fluxions or celerities, we are told that $z.\ z.\ z.\ \dot{z}.\ \ddot{z}.\ \dddot{z}.$* represents a series of quantities, whereof each subsequent quantity is the fluxion of the preceding; and each foregoing is a fluent quantity having the following one for its fluxion?

XLVI. Divers series of quantities and expressions, geometrical and algebraical, may be easily conceived, in lines, in surfaces, in species, to be continued without end or limit. But it will not be found so easy to conceive a series, either of mere velocities, or of mere nascent increments, distinct therefrom and corresponding thereunto. Some perhaps may be led to think the author intended a series of ordinates, wherein each ordinate was the fluxion of the preceding and fluent of the following, i. e. that the fluxion of one ordinate was itself the ordinate of another curve; and the fluxion of this last ordinate was the ordinate of yet another curve; and so on *ad infinitum*. But who can conceive how the fluxion (whether velocity or nascent increment) of an ordinate? Or more than that each preceding quantity or fluent is related to its subsequent or fluxion, as the area of curvilinear figure to its ordinate; agreeably to what the author remarks, that each preceding quantity in such series is as the area of a curvilinear figure, whereof the absciss is $z$, and the ordinate is the following quantity.

XLVII. Upon the whole it appears that the celerities are dismissed, and instead thereof areas and ordinates are introduced. But however expedient such analogies or such expressions may be found for facilitating the modern quadratures, yet we shall not find any light given us thereby into the original real nature of fluxions; or that we are enabled to frame from thence just ideas of fluxions considered in themselves. In all this the general ultimate drift of the author is very clear; but his principles are obscure. But perhaps those theories of the great author are not minutely considered or canvassed by his disciples: who seem eager, as was before hinted, rather to operate than to know,

* De Quadraturâ Curvarum.

rather to apply his rules and his forms, than to understand his principles and enter into his notions. It is nevertheless certain, that in order to follow him in his quadratures, they must find fluents from fluxions; and in order to this, they must know to find fluxions from fluents: and in order to find fluxions, they must first know what fluxions are. Otherwise they proceed without clearness and without science. Thus the direct method precedes the inverse, and the knowledge of the principles is supposed in both. But as for operating according to rules, and by the help of general forms, whereof the original principles and reasons are not understood, this is to be esteemed merely technical. Be the principles therefore ever so abstruse and metaphysical, they must be studied by whoever would comprehend the doctrine of fluxions. Nor can any geometrician have a right to apply the rules of the great author, without first considering his metaphysical notions whence they were derived. These, how necessary soever in order to science, which can never be attained without a precise, clear, and accurate conception of the principles, are nevertheless by several carelessly passed over; while the expressions alone are dwelt on and considered and treated with great skill and management, thence to obtain other expressions by methods suspicious and indirect (to say the least), if considered in themselves, however recommended by induction and authority; two motives which are acknowledged sufficient to beget a rational faith and moral persuasion, but nothing higher.

XLVIII. You may possibly hope to evade the force of all that hath been said, and to screen false principles and inconsistent reasonings, by a general pretence that these objections and remarks are metaphysical. But this is a vain pretence. For the plain sense and truth of what is advanced in the foregoing remarks, I appeal to the understanding of every unprejudiced, intelligent reader. To the same I appeal, whether the points remarked upon are not most incomprehensible metaphysics. And metaphysics not of mine, but your own. I would not be understood to infer, that your notions are false or vain because they are metaphysical. Nothing is either true or false for that reason. Whether a point be called metaphysical or no, avails little. The question is, whether it be clear or obscure, right or wrong, well or ill deduced?

XLIX. Although momentaneous increments, nascent and evanescent quantities, fluxions and infinitesimals of all degrees, are in truth such shadowy entities, so difficult to imagine or conceive distinctly, that (to say the least) they cannot be admitted as principles or objects of clear and accurate science: and although this obscurity and incomprehensibility of your metaphysics had been alone sufficient to allay your pretensions to evidence; yet it hath, if I mistake not, been further shown, that your inferences are no more just than your conceptions are clear, and that your logics are as exceptionable as your metaphysics. It should seem therefore upon the whole, that your conclusions are not attained by just reasoning from clear principles; consequently, that the employment of modern analysts, however useful in mathematical calculations and constructions, doth not habituate and qualify the mind to apprehend clearly and infer justly; and consequently, that you have no right in virtue of such habits to dictate out of your proper sphere, beyond which your judgment is to pass for no more than that of other men.

L. Of a long time I have suspected, that these modern analytics were not scientifical, and gave some hints thereof to the public about twenty-five years ago. Since which time, I have been diverted by other occupations, and imagined I might employ myself better than in deducing and laying together my thoughts on so nice a subject. And though of late I have been called upon to make good my suggestions; yet as the person who made this call doth not appear to think maturely enough to understand, either those metaphysics which he would refute, or mathematics which he would patronize, I should have spared myself the trouble of writing for his conviction. Nor should I now have troubled you or myself with this address, after so long an intermission of these studies, were it not to prevent, so far as I am able, your imposing on yourself and others, in matters of much higher moment and concern. And to the end that you may more clearly comprehend the force and design of the foregoing remarks, and pursue them still further in your own meditations, I shall subjoin the following Queries.

Query 1. Whether the object of geometry be not the proportions of assignable extensions? And whether there be any need of considering quantities either infinitely great or infinitely small?

Qu. 2. Whether the end of geometry be not to measure assignable finite extension? And whether this practical view did not first put men on the study of geometry?

Qu. 3. Whether the mistaking the object and end of geometry hath not created needless difficulties, and wrong pursuits in that science?

Qu. 4. Whether men may properly be said to proceed in a scientific method, without clearly conceiving the object they are conversant about, the end proposed, and the method by which it is pursued?

Qu. 5. Whether it doth not suffice, that every assignable number of parts may be contained in some assignable magnitude? And whether it be not unnecessary, as well as absurd, to suppose that finite extension is infinitely divisible?

Qu. 6. Whether the diagrams in a geometrical demonstration are not to be considered as signs of all possible finite figures, of all sensible and imaginable extensions or magnitudes of the same kind?

Qu. 7. Whether it be possible to free geometry from insuperable difficulties and absurdities, so long as either the abstract general idea of extension, or absolute external extension be supposed its true object?

Qu. 8. Whether the notions of absolute time, absolute place, and absolute motion be not most abstractedly metaphysical? Whether it be possible for us to measure, compute, or know them?

Qu. 9. Whether mathematicians do not engage themselves in disputes and paradoxes, concerning what they neither do nor can conceive? And whether the doctrine of forces be not a sufficient proof of this?*

Qu. 10. Whether in geometry it may not suffice to consider assignable finite magnitude, without concerning ourselves with infinity? And whether it would not be righter to measure large polygons having finite sides, instead of curves, than to suppose curves are polygons of infinitesimal sides, a supposition neither true nor conceivable?

Qu. 11. Whether many points, which are not readily assented to, are not nevertheless true? And whether those in the two following queries may not be of that number?

Qu. 12. Whether it be possible, that we should have had an idea or notion of extension prior to motion? Or whether, if a man had never perceived motion, he would ever have known or conceived one thing to be distant from another?

Qu. 13. Whether geometrical quantity hath coexistent parts? And whether all quantity be not in a flux as well as time and motion?

Qu. 14. Whether extension can be supposed an attribute of a being immutable and eternal?

Qu. 15. Whether to decline examining the principles, and unravelling the methods used in mathematics, would not show a bigotry in mathematicians?

Qu. 16. Whether certain maxims do not pass current among analysts, which are shocking to good sense? And whether the common assumption, that a finite quantity divided by nothing is infinite, be not of this number?

Qu. 17. Whether the considering geometrical diagrams absolutely or in themselves, rather than as representatives of all assignable magnitudes or figures of the same kind, be not a principal cause of the supposing finite extension infinitely divisible; and of all the difficulties and absurdities consequent thereupon?

Qu. 18. Whether from geometrical propositions being general,

* See the treatise Concerning Motion.

and the lines in diagrams being therefore general substitutes or representatives, it doth not follow that we may not limit or consider the number of parts into which such particular lines are divisible?

Qu. 19. When it is said or implied, that such a certain line delineated on paper contains more than any assignable number of parts, whether any more in truth ought to be understood, than that it is a sign indifferently representing all finite lines, be they ever so great: in which relative capacity it contains, i. e. stands for more than any assignable number of parts? And whether it be not altogether absurd to suppose a finite line, considered in itself or in its own positive nature, should contain an infinite number of parts?

Qu. 20. Whether all arguments for the infinite divisibility of finite extension do not suppose and imply, either general abstract ideas or absolute external extension to be the object of geometry? And therefore, whether, along with those suppositions, such arguments also do not cease and vanish?

Qu. 21. Whether the supposed infinite divisibility of finite extension hath not been a snare to mathematicians, and a thorn in their sides? And whether a quantity infinitely diminished, and a quantity infinitely small, are not the same thing?

Qu. 22. Whether it be necessary to consider velocities of nascent or evanescent quantities, or moments, or infinitesimals? And whether the introducing of things so inconceivable be not a reproach to mathematics?

Qu. 23. Whether inconsistencies can be truths? Whether points repugnant and absurd are to be admitted upon any subject, or in any science? And whether the use of infinites ought to be allowed, as a sufficient pretext and apology for the admitting of such points in geometry?

Qu. 24. Whether a quantity be not properly said to be known, when we know its proportion to given quantities? And whether this proportion can be known, but by expressions or exponents, either geometrical, algebraical, or arithmetical? And whether expressions in lines or species can be useful, but so far forth as they are reducible to numbers?

Qu. 25. Whether the finding out proper expressions or notations of quantity be not the most general character and tendency of the mathematics? And arithmetical operation that which limits and defines their use?

Qu. 26. Whether mathematicians have sufficiently considered the analogy and use of signs? And how far the specific limited nature of things corresponds thereto?

Qu. 27. Whether because, in stating a general case of pure algebra, we are at full liberty to make a character denote either a positive or a negative quantity, or nothing at all, we may therefore, in a geometrical case, limited by hypotheses and reasonings, from particular properties and relations of figures, claim the same license?

Qu. 28. Whether the shifting of the hypothesis, or (as we may call it) the *fallacia suppositionis*, be not a sophism, that far and wide infects the modern reasonings, both in the mechanical philosophy and in the abstruse and fine geometry?

Qu. 29. Whether we can form an idea or notion of velocity distinct from and exclusive of its measures, as we can of heat distinct from and exclusive of the degrees on the thermometer, by which it is measured? And whether this be not supposed in the reasonings of modern analysts?

Qu. 30. Whether motion can be conceived in a point of space? And if motion cannot, whether velocity can? And if not, whether a first or last velocity can be conceived in a mere limit, either initial or final, of the described space?

Qu. 31. Where there are no increments, whether there can be any ratio of increments? Whether nothings can be considered as proportional to real quantities? Or whether to talk of their proportions be not to talk nonsense? Also in what sense we are to understand the proportion of a surface to a line, of an area to an ordinate? And whether species or numbers, though properly expressing quantities which are not homogeneous, may yet be said to express their proportion to each other?

Qu. 32. Whether, if all assignable circles may be squared, the circle is not, to all intents and purposes, squared as well as the parabola? Or whether a parabolical area can in fact be measured more accurately than a circular?

Qu. 33. Whether it would not be righter to approximate fairly, than to endeavour at accuracy by sophisms?

Qu. 34. Whether it would not be more decent to proceed by trials and inductions, than to pretend to demonstrate by false principles?

Qu. 35. Whether there be not a way of arriving at truth, although the principles are not scientific, nor the reasoning just? And whether such a way ought to be called a knack or a science?

Qu. 36. Whether there can be science of the conclusion, where there is not evidence of the principles? And whether a man can have evidence of the principles, without understanding them? And therefore whether the mathematicians of the present age act like men of science, in taking so much more pains to apply their principles, than to understand them?

Qu. 37. Whether the greatest genius wrestling with false principles may not be foiled? And whether accurate quadratures can be obtained without new *postulata* or assumptions? And if not, whether those which are intelligible and consistent ought not to be preferred to the contrary? See Sect. XXVIII. and XXIX.